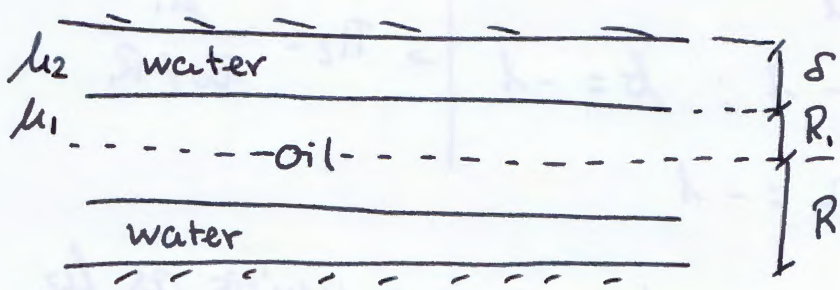
AFM - Homework 3

Sònia Garrido Ballart

(1)



$\delta = R - R_1$

$\rho_1 = \rho_2 = \rho$

$\sigma \rightarrow$ interfacial tension

$\bar{v}_0 = \dfrac{Q_{oil}}{\pi R_1^2}$ ~ avarage velocity of the oil through the pipe

▷ Dimensional analysis

$$\frac{\Delta P}{L} = f(\rho, \bar{v}_0, R, R_1, \mu_1, \mu_2, \sigma)$$

	ρ	$\bar{v}_0$	R	R_1	μ_1	μ_2	σ	$\Delta P/L$
M	1	0	0	0	1	1	1	1
L	-3	1	1	1	-1	-1	0	-2
T	0	-1	0	0	-1	-1	-2	-2

$$M^0 L^0 T^0 = (L T^{-1})^a L^b (M L^{-3})^c M L^{-2} T^{-2}$$

$0 = c + 1;\ c = -1$

$0 = a + b - 3c - 2;\ 0 = -2 + b - 3(-1) - 2;\ b = +1$

$0 = -a - 2;\ a = -2$

$\Pi \quad \Delta P \quad R_1$

$M^0 L^0 T^0 = (LT^{-1})^a L^b (ML^{-3})^c \, ML^{-1}T^{-1}$

$0 = c + 1; \quad c = -1$

$0 = a + b - 3c - 1; \quad b = -1 \quad \rightarrow \Pi_2 = \dfrac{\mu_1}{\bar{v}_0 \rho R_1}$

$0 = -a - 1; \quad a = -1$

$\Pi_3 = \dfrac{\mu_2}{\bar{v}_0 \rho R_1}$ → has the same units as μ_2.

$M^0 L^0 T^0 = (LT^{-1})^a L^b (ML^{-3})^c \, M^0 L^1 T^0$

$a = 0, \quad c = 0; \quad b = -1$

$\Pi_4 = \dfrac{R}{R_1}$

$M^0 L^0 T^0 = (LT^{-1})^a L^b (ML^{-3})^c \, M L^0 T^{-2}$

$0 = c + 1; \quad c = -1$

$0 = a + b - 3c; \quad b = -1$

$0 = -a - 2; \quad a = -2$

$\Pi_5 = \dfrac{\sigma}{\bar{v}_0^2 \rho R_1}$

We obtained Reynolds number for Π_2 and Π_3 and a geometrical ratio for Π_4 and Weber number for Π_5 where we have a relation between the inertia and the capillarity

$\bar{v}_0^2 \rho R_1$

weber number gives the relative importance of the inertia compared to the surface tension, this is important to determine if waves will develop. Under flowing conditions, surface tension is the stabilizing force which resist the formation of curved interfaces. An appropiate inequality in order for waves not to form would be:

$$We \ll 1$$

we have not considered gravity as a diving force to form waves at the interface as there is no density contrast to drive density waves at interface ($\Delta \rho = \rho_1 - \rho = 0$)

we assume that the flow in the pipe remains a perfect smooth core annular flow and that the pressure change across the interface is negligible and pressure gradient is

$$\frac{\partial P}{\partial z} = -\frac{\Delta P}{L} \quad r \in [0, R]$$

steady state velocity field is $v = (0, 0, v_z(r))$

Navier - Stokes equation

$$-\frac{\partial p}{\partial z} + \frac{\mu_1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = 0 \qquad 0 \leq r \leq R_1$$

$$-\frac{\partial p}{\partial z} + \frac{\mu_2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = 0 \qquad R_1 \leq r \leq R$$

$$v_z^{(1)} = \frac{1}{4\mu_1}\left(\frac{\Delta P}{L}\right)[R_1^2 - r^2] + \frac{1}{4\mu_2}\left(\frac{\Delta P}{L}\right)[R^2 - R_1^2]; \; 0 \leq r \leq R_1$$

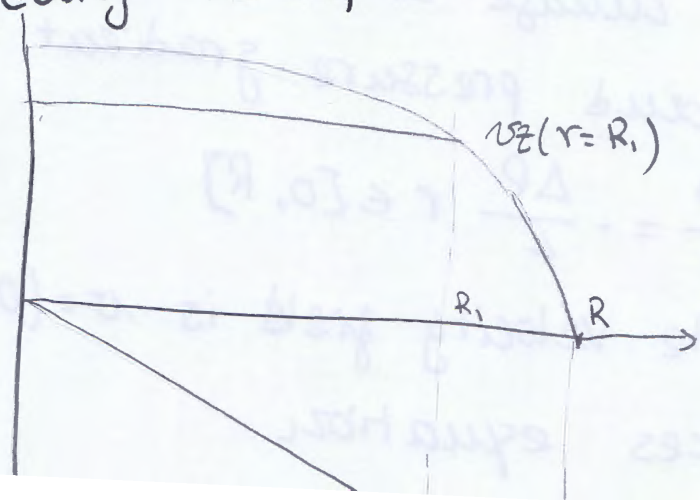
$$v_z^{(2)} = \frac{1}{4\mu_2}\left(\frac{\Delta P}{L}\right)[R^2 - r]; \; R_1 \leq r \leq R$$

At the interface $v_z^{(1)}(r = R_1) = v_z^{(2)}(r = R_1)$

The shear stress is negative and increases linearly in magnitude across the entire pipe.

$$\tau = -\frac{r}{2}\left(\frac{\Delta P}{L}\right)$$

The velocity field is continuous but changes slope by a factor of (μ_1/μ_2) at the core/shell boundary $r = R_1$



The volumen flow rate of oil Q_θ is given by:

$$Q_\theta = \int_0^{R_1} 2\pi v_z^{(1)}\, dr = \frac{\pi}{2}\,\frac{\Delta P}{L}\int_0^{R_1}\left(\frac{r}{\mu_1}\left[R_1^2 - r^2\right] + \right.$$

$$\left. + \frac{r}{\mu_2}\left[R^2 - R_1^2\right]\right) dr = \frac{\pi}{2}\,\frac{\Delta P}{L}\left[\frac{R_1^4}{4\mu_1} + \frac{R_1^4}{2\mu_2}\left(R^2 - R_1^2\right)\right]$$

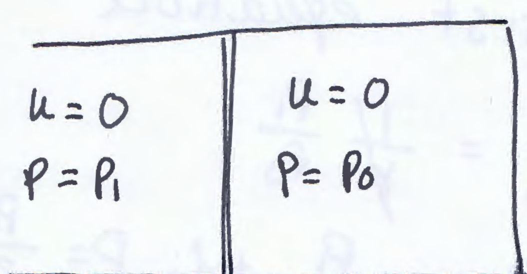
$$Q_w = \int_{R_1}^{R} 2\pi v_z^{(2)}\, dr = \int_{R_1}^{R} 2\pi \frac{1}{4\mu_2}\left(\frac{\Delta P}{L}\right)\left[R^2 - r\right] dr =$$

$$= \frac{\pi}{8\mu_2}\left(\frac{\Delta P}{L}\right)\left[R^2 - R_1^2\right]^2 \rightarrow \text{volume flow}$$

rate of water.

---

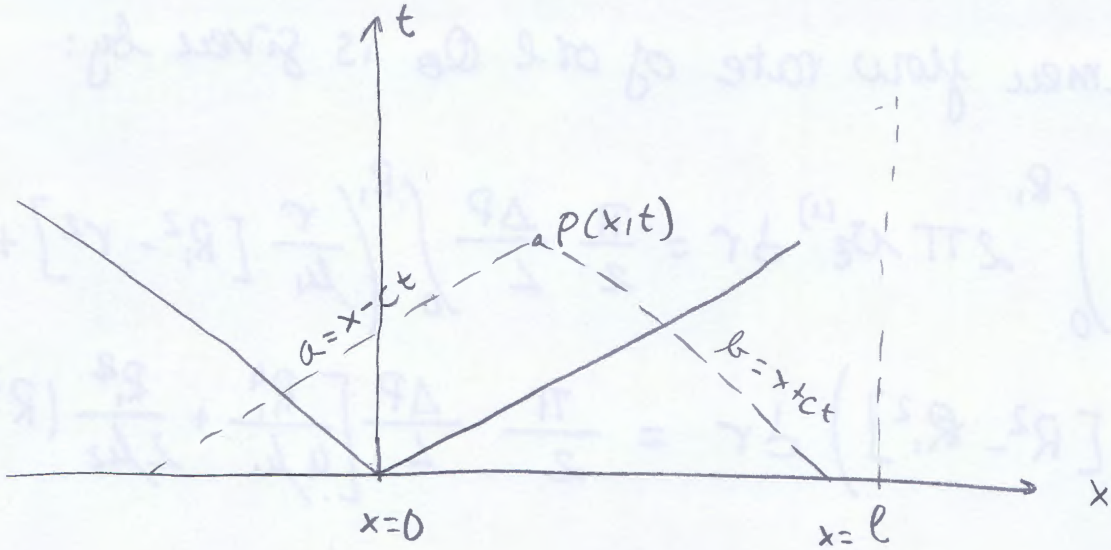
(2)



$P_1 > P_0$

$(P_1 - P_0)/P_0 \ll 1$

The wave patter which results from the breaking of the diaphragm would be:



using the Reimann invariants

$a = x - ct \rightarrow$ constant

$$\frac{u}{c} + \frac{1}{\gamma}\frac{P}{P_0} = 0 + \frac{1}{\gamma}\frac{P_1}{P_0}$$

$b = x + ct \rightarrow$ constant

$$\frac{u}{c} - \frac{1}{\gamma}\frac{P}{P_0} = 0 - \frac{1}{\gamma}\frac{P_0}{P_0} \ ; \quad \frac{u}{c} = \frac{1}{\gamma}\left(\frac{P}{P_0} - 1\right)$$

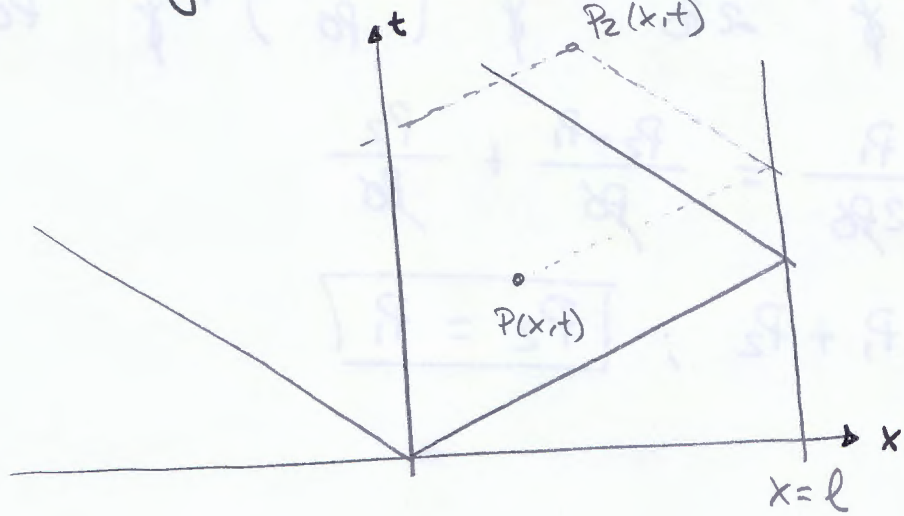
Substituting in the first equation

$$\frac{\cancel{1}}{\cancel{\gamma}}\left(\frac{P}{P_0} - 1\right) + \frac{\cancel{1}}{\cancel{\gamma}}\frac{P}{P_0} = \frac{\cancel{1}}{\cancel{\gamma}}\frac{P_1}{P_0}$$

$$\frac{2P}{P_0} - 1 = \frac{P_1}{P_0} \ ; \quad \frac{2P}{P_0} = \frac{P_1}{P_0} + 1; \quad P = \frac{P_0}{2}\left(\frac{P_1}{P_0} + 1\right)$$

$$P = \frac{P_1 + P_0}{2}$$

▷ Wave reflection



$$a = x - ct \rightarrow cte$$

$$\cancel{\frac{u}{c}}^{0} + \frac{1}{\gamma}\frac{P_\ell}{P_0} = \frac{1}{\gamma}\left(\frac{P_0+P_1}{2P_0} - 1\right) + \frac{1}{\gamma}\frac{P_0+P_1}{2P_0}$$

$$\frac{P_\ell}{P_0} = \frac{1}{2P_0}\left(P_1 - P_0\right) + \frac{P_0+P_1}{2P_0}$$

$$P_\ell = \frac{1}{2}\left(P_1 - P_0\right) + \frac{P_0+P_1}{2} = P_1 \rightarrow \text{pressure on } x=\ell$$

The solution in $p_2(x,t)$ can be computed by following the characteristic lines.

$$a = x - ct \rightarrow \frac{1}{2\gamma}\left(\frac{P_1-P_0}{P_0}\right) + \frac{1}{\gamma}\frac{P_0+P_1}{2P_0} = \frac{u}{c} + \frac{1}{\gamma}\frac{P_2}{P_0}$$

$$b = x + ct \rightarrow \cancel{\frac{u}{c}}^{0} - \frac{1}{\gamma}\frac{\overset{\rightarrow P_1}{P_\ell}}{P_0} = \frac{u}{c} - \frac{1}{\gamma}\frac{P_2}{P_0}$$

$$\frac{u}{c} = \frac{1}{\gamma}\left(\frac{P_2-P_1}{P_0}\right)$$

$$\frac{1}{2\cancel{\gamma}}\left(\frac{P_1-\cancel{P_0}}{P_0}\right)+\frac{1}{\cancel{\gamma}}\,\frac{\cancel{P_0}+P_1}{2P_0}=\frac{1}{\cancel{\gamma}}\left(\frac{P_2-P_1}{P_0}\right)+\frac{1}{\cancel{\gamma}}\,\frac{P_2}{P_0}$$

$$\frac{P_1}{2\cancel{P_0}}+\frac{P_1}{2\cancel{P_0}}=\frac{P_2-P_1}{\cancel{P_0}}+\frac{P_2}{\cancel{P_0}}$$

$$P_1 = P_2 - P_1 + P_2 \quad ; \quad \boxed{P_2 = P_1}$$