Advanced Fluid Mechanics Homework 4
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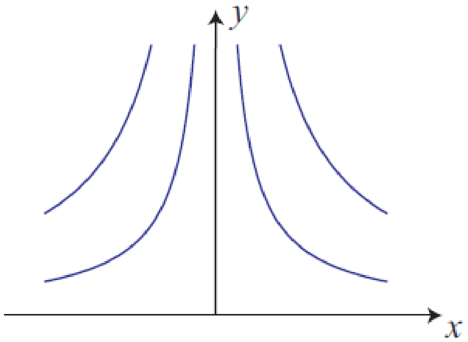
1. Consider a fluid stream whose velocity vector coincides with the y-axis that impinges on a plane boundary that coincides with the x-axis, as shown in the figure below.



(a) If we consider an ideal fluid, velocity can be obtained using the following stream function

$$\psi(r,\theta) = Ur^2 \sin(2\theta)$$

Compute the velocity field in Cartesian coordinates (u; v) and show that it verifies the boundary conditions. Obtain an expression for the pressure distribution.

$$\begin{cases} U_r = nUR^{n-1}cos_{n\theta} \\ U_r = -nUR^{n-1}sin_{n\theta} \end{cases}$$

Using n=2 because in the case shown in the figure we have π/2:

$$\begin{cases} U_r = 2URcos_{2\theta} \\ U_r = -2URsin_{2\theta} \end{cases}$$

So changing to x and y, the next velocity field is obtained:

$$\begin{cases} u = 2Ux \\ v = -2Uy \end{cases}$$

Then from the Bernoulli equation:

$$\int_1^2 \frac{\partial u}{\partial t} d\psi + \left(\frac{1}{2}u_2^2 + \frac{p_0}{\rho} - F_2\right) - \left(\frac{1}{2}u_1^2 + \frac{p}{\rho} - F_1\right) = 0$$

$$0 + (\rho 2Uy^2 + p_0 - 0) - (\rho 2Ux^2 + p - 0) = 0$$

So the expression for the pressure distribution obtained is:

$$p = p_0 - 2\rho U(x^2 + y^2)$$

(b) Show that the former velocity and pressure distributions verify the Navier-Stokes equations but not the boundary conditions for the viscous problem.

A potential flow has to follow:

$$u = \nabla \emptyset$$

Then it is easy to compute:

$$\nabla^2 u = \nabla^2(\nabla\emptyset) = \nabla(\nabla^2\emptyset) = 0$$

So the velocity field found in the previous questions fulfils the Navier-Stokes equations for viscous problems.

We can observed that the potential flow used do not respect the no-slip condition necessary.

(c) A solution for the viscous problem can be obtained modifying the potential flow in such a way that meeting the boundary condition would be possible. If we attempt

$$u = 2Uxf'(y)$$

show that the continuity equation requires that :

$$v = -2Uf(y)$$

State appropriate boundary conditions for the function *f*.

If the horitzontal component of velocity is defined as $u = 2Uxf'(y)$ then the continuity requires:

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -2Uf'(y)$$

And integrating, the vertical component of the velocity:

$$v = -2Uf(y)$$

The boundary conditions has to respect the behaviour of the flow far of the stagnation point but also near it. Therefore, they are modified and described in the next way:

$$u(x,0) = 0 \rightarrow f'(0) = 0$$
$$v(x,0) = 0 \rightarrow f'(0) = 0$$
$$f'(y) \rightarrow 1$$
$$y \rightarrow \infty$$

(d) Use the y-momentum equation to obtain an expression for the pressure distribution in terms of the function *f*. In order to completely determine the pressure distribution, you can use that for large values of y; the potential flow pressure should be recovered.

y-momentum:

$$4U^2 ff' = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2U\text{v}f''$$

Integrating respect$\frac{dy}{dp}$:

$$2U^2f^2 + 2U\nu f' + g(x) = -\frac{1}{\rho}p$$

Where the term g(x) will correspond with the behaviour of the pressure far from the stagnation point, and correspond with that expression for the pressure. Then with all this, we can write the next expression for the pressure:

$$p(x,y) = p_0 - 2\rho U^2 f^2 + 2\rho U \text{v}(1-f') - 2\rho U^2 x^2$$

(e) Using the x-momentum equation and the pressure distribution obtained in the previous point, obtain a differential equation for the function *f*. Show that the problem can be solved using the boundary conditions stated in point c).

Using the equation of x-momentum:

$$4U^2 x f'^2 - 4U^2 x f f'' = -\frac{1}{\rho}\frac{dp}{dx} + 2U\text{v}x f'''$$

Substituting $\frac{dp}{dx} = -4\rho U^2 x$, we obtain:

$$4U^2 x f'^2 - 4U^2 x f f'' = -4U^2 x + 2U\text{v}x f'''$$

And writing the higher terms in the left side:

$$\frac{v}{2U} x f''' + f f'' - f'^2 + 1 = 0$$

2. Use the Kármán-Pohlhausen approximation to compute the boundary layer solution for a uniform flow over a flat plate. Assume a quadratic polynomial form for the velocity profile:

$$\frac{u}{U} = a + b\frac{y}{\delta} + c\left(\frac{y}{\delta}\right)^2$$

And use the following boundary conditions:

$$u = 0 \text{ at } y = 0$$
$$u = U,\ \frac{\partial u}{\partial y} = 0 \text{ at } y = \delta$$

Compare the results with the exact Blasius solution and with the ones obtained assuming a cubic velocity profile.

Using Kármán-Pohlhausen approximation, we have the next two expressions:

$$\begin{cases} U^2\theta = \int_0^\infty u(U-u)dy \\ \dfrac{d}{dx}(U^2\theta) = \dfrac{\tau_0}{\rho} \end{cases}$$

And it can be obtained:

$$\frac{d}{dx}\int_0^{\infty} u(U-u)dy = \frac{\tau_0}{\rho}$$

So, with that expression below and the boundary conditions it is possible to compute a,b,c and obtained the next expression:

$$\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$$

And integrating :

$$\int_0^{\delta} u(U-u)dy,$$

$$\delta = 0.84\sqrt{\frac{x\nu}{U}}$$

With the same procedure but using a cubic velocity profile, the next expression is obtained:

$$\delta = 4.64\sqrt{\frac{x\nu}{U}}$$

And the Blasius exact solution:

$$\delta = 5\sqrt{\frac{x\nu}{U}}$$

So comparing the expression obtained with the other two we can see that using a quadratic profile for the same parameters the δ will be smaller and far from the exact solution.