ADVANCE FLUID MECHANICS 30/12/15

ASSIGNMENT-4

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(1)

Given stream function:

$$\psi(r,\theta) = Ur^2 \sin(2\theta)$$

$$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = Ur2\cos 2\theta = 2Ur\cos 2\theta$$

$$V_\theta = \frac{-\partial \psi}{\partial r} = -2Ur\sin 2\theta$$

$$r = \sqrt{x^2+y^2}, \quad \theta = \tan^{-1}(y/x)$$

$$u = V_r\cos\theta - V_\theta \sin\theta$$

$$u = 2U_r\cos 2\theta\cos\theta + 2U_r\sin 2\theta\sin\theta$$

$$u = 2U_r[\cos 2\theta\cos\theta + \sin 2\theta\sin\theta]$$

$$= 2U_r[\cos^3\theta + \sin^2\theta\cos\theta]$$

$$= 2U_r\cos\theta = 2U\sqrt{x^2+y^2}\cos(\tan^{-1}(y/x))$$

$$= 2U\sqrt{x^2+y^2} \times \frac{y}{x\sqrt{x^2+1}}$$

$$\therefore \boxed{u = 2Ux}$$

$$v = V_r\sin\theta + V_\theta\cos\theta.$$

$$= 2U_r\cos 2\theta\sin\theta - 2U_r\sin 2\theta\sin\theta$$

$$= 2U_r[-\sin\theta\cos 2\theta - \sin^3\theta]$$

$$v = -2U_r\sin\theta = -2U\sqrt{x^2+y^2}\sin(\tan^{-1}(y/x))$$

$$\boxed{v = -2Uy}$$

$$\therefore \quad \boxed{\begin{aligned} u &= 2Ux \\ v &= -2Uy \end{aligned}} \begin{aligned} &\longrightarrow (1) \\ &\longrightarrow (2) \end{aligned}$$

Boundary conditions :- [IDEAL]

at $y = 0$; $u \neq 0$ but $v = 0$

at $y = \infty$; $u = 0$ but $v = y = -v$ [Because the flow is down-wards]

Put $y = 0$ in Eqn (1), we get $u = 2Ux$

Put $y = 0$ in Eqn (2), we get $v = 0$

Put $y = \infty$ in Eqn (1), we get $u = 0$

Put $y = \infty$ in Eqn (2), we get $v = -v$

Expression for Pressure Distribution :-

Applying Bernoulli's Equation at a stagnant point and at a point x, we have

$$P_0 + 0 = p + \rho\left(\frac{u^2 + v^2}{2}\right)$$

$$P_0 = p + \rho\left(\frac{4U^2x^2 + 4U^2y^2}{2}\right)$$

$$P_0 = p + 2\rho U^2(x^2 + y^2)$$

$$\Rightarrow \quad \boxed{p = P_0 - 2\rho U^2(x^2 + y^2)} \longrightarrow (3)$$

(b) Continuity Equation :- $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

$\Rightarrow 2U - 2U = 0$ satisfied

<u>X-momentum :</u>

$$u\frac{\partial u}{\partial x} + \cancel{v\frac{\partial u}{\partial y}}^{0} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \cancel{\frac{\partial^2 u}{\partial y^2}}^{0}\right]$$

$$2Ux2U = \frac{-1}{\cancel{\rho}}(-4\cancel{\rho}U^2x) + 0$$

$$\Rightarrow 4U^2x = 4U^2x \quad [\text{Verified}]$$

<u>Y-momentum :</u>

$$\cancel{u\frac{\partial v}{\partial x}}^{0} + v\frac{\partial v}{\partial y} = \frac{-1}{\rho}\frac{\partial P}{\partial y} + \nu\left[\cancel{\frac{\partial^2 v}{\partial x^2}}^{0} + \cancel{\frac{\partial^2 v}{\partial y^2}}^{0}\right]$$

$$-4U^2y = \frac{1}{\cancel{\rho}}(-4\cancel{\rho}U^2y) \quad (\text{Verified})$$

But for the viscous problem, as friction is significant, so no slip Boundary condition will come into the play. Therefore at $y = 0$; $u = 0$ and $v = 0$ and at $y = \infty$; $u = 0$ and $v = -V$

$\therefore$ They won't satisfy the B.C mentioned previously.

(c) Given that $u = 2Ux f'(y)$

from Continuity Eqn, we have

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$2Uf'(y) + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial v}{\partial y} = -2Uf'(y)$$

On Integrating,

$$\boxed{v = -2Uf(y)}$$

Appropriate Boundary Conditions :- (viscous)

$u(x,0) = 0$, $f(y) = 0$

$v(x,0) = 0$, $f(y) = 0$

If $y \to \infty$ $\Rightarrow$ $f(y) = y$

$f'(y) = 1$

(d) y- momentum Equation :-

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right]$$

$$2Uxf'(y)\times 0 + 2Uf(y)\,2Uf'(y) = -\frac{1}{\rho}\frac{\partial P}{\partial y} - 2U\nu f''(y)$$

$$\therefore\ 4U^2 f(y) f'(y) = -\frac{1}{\rho}\frac{\partial P}{\partial y} - 2U\nu f''(y)$$

$$\frac{\partial P}{\partial y} = -4\rho U^2 f(y) f'(y) + 2U\nu f''(y)\rho$$

On Integrating, we have

$$\partial P = [-4\rho U^2 f(y) f'(y) - 2U\nu f''(y)\rho]\, \partial y$$

$$\boxed{P(x,y) = -2\rho U^2 (f)^2 - 2\rho U\nu f' + g(x)}$$

We know that $g(x)$ is a function of x.
That can be determined by comparison with potential flow pressure distribution that should be reconsidered with large values of y.

$f(y) \to y$ for large values of y.

$$\boxed{P(x,y) = -2\rho U^2 y^2 - 2\rho U\nu + g(x)} \to (4)$$

Comparing (3) & (4), we have

$$P_0 - 2\rho U^2 (x^2 + y^2) = -2\rho U^2 y^2 - 2\rho U\nu + g(x)$$

$$P_0 - 2\rho U^2 x^2 - 2\rho \cancel{U^2 y^2} = -2\rho \cancel{U^2} y^2 - 2\rho U\nu + g(x)$$

$$\therefore \boxed{g(x) = P_0 - 2\rho U^2 x^2 + 2\rho U\nu} \to (5)$$

Use (5) in (4), we have

$$\boxed{P(x,y) = -2\rho U^2 y^2 - 2\rho U\nu + P_0 - 2\rho U^2 x^2 + 2\rho U\nu} \to (6)$$

(e) X- Momentum:-

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

$$2Ux\, f'(y)\, 2Uf'(y) + \left[-2Uf(y) \times 2Ux\, f'(y)\right] = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left[0 + 2Ux\, f'''(y)\right]$$

$$4U^2x\, f'(y) - 4U^2x\, f'(y) f(y) = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left[2Ux\, f'''(y)\right] \rightarrow (7)$$

From Eqn (6):-

$$\frac{\partial P}{\partial x} = -4\rho U^2 x$$

using this equation in (7), we have

$$4U^2x\left[f'(y)\right]^2 - 4U^2x\, f'(y) f(y) = 4U^2x + \nu 2Ux\, f'''(y)$$

Now, divide throughout by $4U^2x$

$$f'(y)^2 - f'(y)f(y) = 1 + \frac{\nu}{2U} f'''(y)$$

$$\boxed{\frac{\nu}{2U} f'''(y) + f(y)f'(y) - f'(y)^2 + 1 = 0} \rightarrow (*)$$

- Boundary Conditions :-

at $u(x,0) = 0 \Rightarrow f'(0) = 0$

at $v(x,0) = 0 \Rightarrow f(0) = 0$

as $y \to \infty \Rightarrow f(y) = y \Rightarrow f'(y) = 1$

Diff. Eqn ⊛ can be solved not only to the governing Equations but also the viscous Boundary conditions provided that $f(y)$ satisfies the above mentioned Boundary conditions.

The above problem can be solved if $\frac{\nu}{2U}$ parameters is free, then the result would be valid for all kinematic viscosity & flow velocities

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2 Given Quadratic polynomial form of the velocity profile.

$$\frac{u}{U} = a + b\frac{y}{\delta} + c\left(\frac{y}{\delta}\right)^2$$

Given Boundary conditions :-

$$\text{at } y = 0, \quad u = 0$$

$$\text{at } y = \delta, \quad u = U$$

$$\text{at } y = \delta, \quad \frac{du}{dy} = 0$$

At $u = 0$

Put $y = 0$, we have

$$\boxed{0 = a}$$

At $u = U$

Put $y = \delta$, we have

$$1 = 0 + b + c$$

$$\boxed{b+c=1}$$

Put $y=\delta$, $\dfrac{\partial u}{\partial y}=0$

$$\frac{1}{U}\frac{\partial u}{\partial y} = \frac{b}{\delta} + \frac{c}{\delta^2}\,2y$$

at. $y=\delta$,

$$0 = \frac{b}{\delta} + \frac{2c}{\delta}$$

$$\Rightarrow \quad \boxed{b+2c=0}$$

$$\Rightarrow \quad \boxed{b=2, \quad c=-1}$$

$$\boxed{\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2}$$

Put $\dfrac{y}{\delta} = \eta$

$dy = \delta\, d\eta$

$$\therefore \quad \frac{u}{U} = 2\eta - \eta^2$$

Now, momentum thickness,

$$\theta = \int_0^{\delta} \frac{u}{U}\left[1-\frac{u}{U}\right) dy$$

$$= \delta\int_0^1 (2\eta-\eta^2)(1-2\eta+\eta^2)\, d\eta$$

$$= \delta\int_0^1 (2\eta-4\eta^2+2\eta^3-\eta^2+2\eta^3-\eta^4)\, d\eta$$

$$= \delta\left[\eta^2 - \frac{4}{3}\eta^3 + \frac{1}{2}\eta^4 - \frac{\eta^3}{3} + \frac{1}{2}\eta^4 - \frac{\eta^5}{5}\right]_0^1$$

$$= \delta\left[1 - \frac{4}{3} + 1 - \frac{1}{3} - \frac{1}{5}\right]$$

$$= \delta\left[2 - \frac{5}{3} - \frac{1}{5}\right]$$

$$\theta = \delta\left[\frac{30 - 25 - 3}{15}\right]$$

$$\boxed{\theta = \frac{2}{15}\delta}$$

The wall stress is given by,

$$\tau_w = \mu \frac{\partial u}{\partial y}\Big|_{y=0}$$

$$\tau_w = \mu\left[\frac{\partial}{\delta \partial \eta}\left\{U(2\eta - \eta^2)\right\}\right]_{\eta=0}$$

$$\tau_w = \frac{\mu}{\delta} U[2 - 2\eta]$$

$$\boxed{\tau_w = \frac{2\mu U}{\delta}}$$

But we have momentum Integral Equation as

$$\frac{d}{dx}(U^2\theta) = \frac{\tau_w}{\rho}$$

$$\frac{\cancel{2}}{15}\left(\frac{d\delta}{dx}\right) = \frac{\cancel{2}\mu U}{\delta \rho U^2}$$

$$\int \delta \, d\delta = \int \frac{15 \mu}{\rho U} \, dx$$

$$\frac{\delta^2}{2} = \frac{15 \nu x}{U} + c$$

at leading edge, $x = 0$; $\delta = 0$

$C = 0$

$$\therefore \frac{\delta^2}{2} = \frac{15 \nu x}{U}$$

$$\delta^2 = \frac{30 \nu x}{U}$$

$\sqrt{30} = 5.4772$

$$\delta = 5.4772 \sqrt{\frac{\nu x^2}{x U}}$$

$$\delta = \frac{5.4772 x}{\sqrt{R_e}}$$

$$\boxed{\frac{\delta}{x}_{(Q)} = \frac{5.4772}{\sqrt{R_e}}}$$

K.P Approximation for Quadratic Polynomial

W.K.T $\frac{\delta}{x}_{(B)} = \frac{5}{\sqrt{R_e}}$ (Blasius Solution)

$$\frac{\delta}{x}_{(C)} = \frac{4.64}{\sqrt{R_e}} \text{ (Cubic Polynomial)}$$

$$\therefore \boxed{\frac{\delta}{x}_{(Q)} > \frac{\delta}{x}_{(B)} > \frac{\delta}{x}_{(C)}}$$