Advance Fluid Mechanics — Home work - (3)

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Problem (1).

(a). Given relative informations to express:

$$\frac{\Delta P}{L} = f(\rho, \bar{v}_0, R, R_1, \mu_1, \mu_2, \sigma)$$

Dimensional matrix of the problem reads.

	$\Delta P/L$	ρ	$\bar{v}_0$	R	R_1	μ_1	μ_2	σ
M	1	1	0	0	0	1	1	1
L	-2	-3	1	1	1	-1	-1	0
T	-2	0	-1	0	0	-1	-1	-2

n = no/. of parameters = 8

r = rank of matrix = 3

$\therefore$ $n - r = 8 - 3 = 5$ π products $[\pi_1, \pi_2, \pi_3, \pi_4, \pi_5]$

As the rank of matrix is 3, three primary variables need to be selected. Following the statement of problem $\bar{v}_0$, R_1 and ρ are selected

For each remaining variable, π-products are defined as

$$\pi_1 = (\Delta P/L)\, \bar{v}_0^{a_1} R_1^{b_1} \rho^{c_1}$$

$$\pi_2 = R\, \bar{v}_0^{a_2} R_1^{b_2} \rho^{c_2}$$

$$\pi_3 = \mu_1\, \bar{v}_0^{a_3} R_1^{b_3} \rho^{c_3}$$

$$\pi_4 = \mu_2\, \bar{v}_0^{a_4} R_1^{b_4} \rho^{c_4}$$

$$\pi_5 = \sigma\, \bar{v}_0^{a_5} R_1^{b_5} \rho^{c_5}$$

$$\pi_1 = (\Delta P/L)\, \bar{V}_0^{a_1} R_1^{b_1} \rho^{c_1}$$

$$M^0L^0T^0 = [ML^{-2}T^{-2}][LT^{-1}]^{a_1}[L]^{b_1}[ML^{-3}]^{c_1}$$

$M^0 \to \quad 1 + c_1 = 0 \quad \Rightarrow \quad \boxed{c_1 = -1}$

$L^0 \to \quad -2 + a_1 + b_1 - 3c_1 = 0 \quad \Rightarrow \quad a_1 + b_1 + 1 = 0$

$T^0 \to \quad -2 - a_1 = 0 \quad \Rightarrow \quad \boxed{a_1 = -2} \quad \& \quad \boxed{b_1 = 1}$

$$\therefore \boxed{\pi_1 = \frac{(\Delta P/L)}{(\bar{V}_0^2\, \rho/R_1)}}$$

$$\pi_2 = R\, \bar{V}_0^{a_2} R_1^{b_2} \rho^{c_2}$$

$$[M^0L^0T^0] = [L][LT^{-1}]^{a_2}[L]^{b_2}[ML^{-3}]^{c_2}$$

$M^0 \to \quad \boxed{c_2 = 0}$

$L^0 \to \quad 1 + a_2 + b_2 - 3c_2 = 0 \quad \Rightarrow \quad a_2 + b_2 + 1 = 0$

$T^0 \to \quad -a_2 = 0 \quad \Rightarrow \quad \boxed{a_2 = 0} \quad \& \quad \boxed{b_2 = -1}$

$$\therefore \boxed{\pi_2 = \frac{R}{R_1}}$$

$$\pi_3 = \mu_1\, \bar{V}_0^{a_3} R_1^{b_3} \rho^{c_3}$$

$$[M^0L^0T^0] = [ML^{-1}T^{-1}][LT^{-1}]^{a_3}[L]^{b_3}[ML^{-3}]^{c_3}$$

$M^0 \to 0 = 1 + c_3 \quad \Rightarrow \quad \boxed{c_3 = -1}$

$L^0 \to 0 = -1 + a_3 + b_3 - 3c_3 \quad \Rightarrow \quad a_3 + b_3 + 2 = 0$

$T^0 \to 0 = -1 - a_3 \quad \Rightarrow \quad \boxed{a_3 = -1} \quad \& \quad \boxed{b_3 = -1}$

$$\therefore \boxed{\pi_3 = \frac{\mu_1}{\bar{V}_0 R_1 \rho}}$$

Similarly, $\boxed{\Pi_4 = \dfrac{\mu_2}{\bar{v}_0 R_1 \rho}}$

$$\Pi_5 = \sigma \, \bar{v}_0^{a_5} R_1^{b_5} \rho^{c_5}$$

$$[M^0 L^0 T^0] = [M T^{-2}] [L T^{-1}]^{a_5} [L]^{b_5} [M L^{-3}]^{c_5}$$

$$M^0 \rightarrow \quad 0 = 1 + c_5 \Rightarrow \boxed{c_5 = -1}$$

$$L^0 \rightarrow \quad 0 = a_5 + b_5 - 3c_5 \Rightarrow a_5 + b_5 + 3 = 0$$

$$T^0 \rightarrow \quad 0 = -2 - a_5 \Rightarrow \boxed{a_5 = -2} \text{ \& } \boxed{b_5 = -1}$$

$$\therefore \boxed{\Pi_5 = \frac{\sigma}{\bar{v}_0^2 R_1 \rho}}$$

And finally, a further simplification of dimensionless parameters applies by taking Π_3/Π_4 :

$$\Pi_1 = \frac{(\Delta P/L)}{(\bar{v}_0^2 \rho/R_1)}, \quad \Pi_2 = \frac{R}{R_1}, \quad \Pi_3 = \frac{\mu_1}{\mu_2}, \quad \Pi_4 = \frac{\bar{v}_0 R_1 \rho}{\mu_1} = Re$$

$$\text{\& } \Pi_5 = \frac{\bar{v}_0^2 R_1 \rho}{\sigma} = We$$

Where the inversion of last 2 parameters of (Π_4 & Π_5) has been done in order to recognize Reynolds Number & Weber Number of oil flow. After all manipulations the new dimensionless form of problem reads.

$$\boxed{\frac{(\Delta P/L)}{(\bar{v}_0^2 \rho/R_1)} = F\left[R/R_1, \mu_1/\mu_2, Re_{R_1}, We_{R_1} \right]}$$

Question (b) :-

As point out in previous question the new dimensionless form of problem depends on the Reynolds & Weber numbers of oil flow. Moreover, the first dimensionless group Π_1 can also be understood as a friction factor.

Among these parameters, the most important in determin-ing whether waves will develop along the interface is the Weber number of oil fluid, We_{R_1}.

According to this definition, interfacial tension is the work which must be expanded to increase size of the interface b/w 2 adjacent phases which do not mix complet-ely with one another.

That means, if no external forces apply, the liquid phases minimize the size of their interface. Therefore, waves will not form unless inertia forces are very small compared to surface tension/equivalently, if the Weber number is very small,

$$\boxed{We_{R_1} << 1}$$

Question (c) :-

Since the fluids are density matched, i.e $\rho_1 = \rho_2 = \rho$, gravity effects can be neglected & it is reasonable to assume that density will not contribute to formation of waves at interface.

Question ⓓ:-

Given the hypothesis & recalling that gravity effects are neglected, let us proceed to write Navier Stokes eq.n in cylindrical co-ordinates for an isotrophic and incompressible Newtonian fluid:

* The continuity and the θ-momentum equation become trivially verified.
* the r-momentum gives that pressure does not depend on r. Thus, $p = p(z)$.
* The z-momentum equation reduces to

$$-\frac{\Delta P}{L} = \frac{\mu_1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right), \quad 0 \le r \le R_1 \tag{1}$$

$$-\frac{\Delta P}{L} = \frac{\mu_2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right), \quad R_1 \le r \le R$$

Regarding Boundary conditions, the no-slip condition must be imposed at the wall and continuity of the velocity field and continuity of tractions must hold along interface. More precisely,

$$v_z(R) = 0, \quad v_z(R_1^-) = v_z(R_1^+) \quad \text{and}$$

$$(\bar{n}\cdot\bar{\sigma})\Big|_{r=R_1^-} = (\bar{n}\cdot\bar{\sigma})\Big|_{r=R_1^+}$$

Assuming the pressure change accross the interface is negligible & writing the symmetric gradient in cylindrical co-ordinates, the condition for tractions simplifies to

-a relation between the shear stresses:

$$\tau_{rz}\Big|_{r=R_1^-} = \mu_1 \frac{dv_z}{dr}\Big|_{r=R_1^-} = \mu_2 \frac{dv_z}{dr}\Big|_{r=R_1^+} = \tau_{rz}\Big|_{r=R_1^+}$$

Question ⓒ :-

Equations (1) have only one unknown $v_z(r)$ and can be straightforwardly integrated. Enforcement of the boundary conditions gives the integration constants. After these steps, the fully developed velocity field $v_z(r)$ has the expression

$$\frac{1}{4\mu_1}\left(\frac{\Delta P}{L}\right)(R_1^2 - r^2) + v_I, \quad 0 \leq r \leq R_1,$$

$$\frac{1}{4\mu_2}\left(\frac{\Delta P}{L}\right)(R^2 - r^2), \quad R_1 \leq r \leq R,$$

Where v_I is the velocity along interface:

$$v_I = \frac{1}{4\mu_2}\left(\frac{\Delta P}{L}\right)(R^2 - R_1^2)$$

Moreover, derivation of $v_z(r)$ in the r-direction allows to find the shear stresses across the pipe, which is always negative and linear:

$$\tau_{rz}(r) = -\frac{r}{2}\left(\frac{\Delta P}{L}\right), \quad 0 \leq r \leq R.$$

The velocity and shear stresses profiles across the pipe is shown in fig ①.

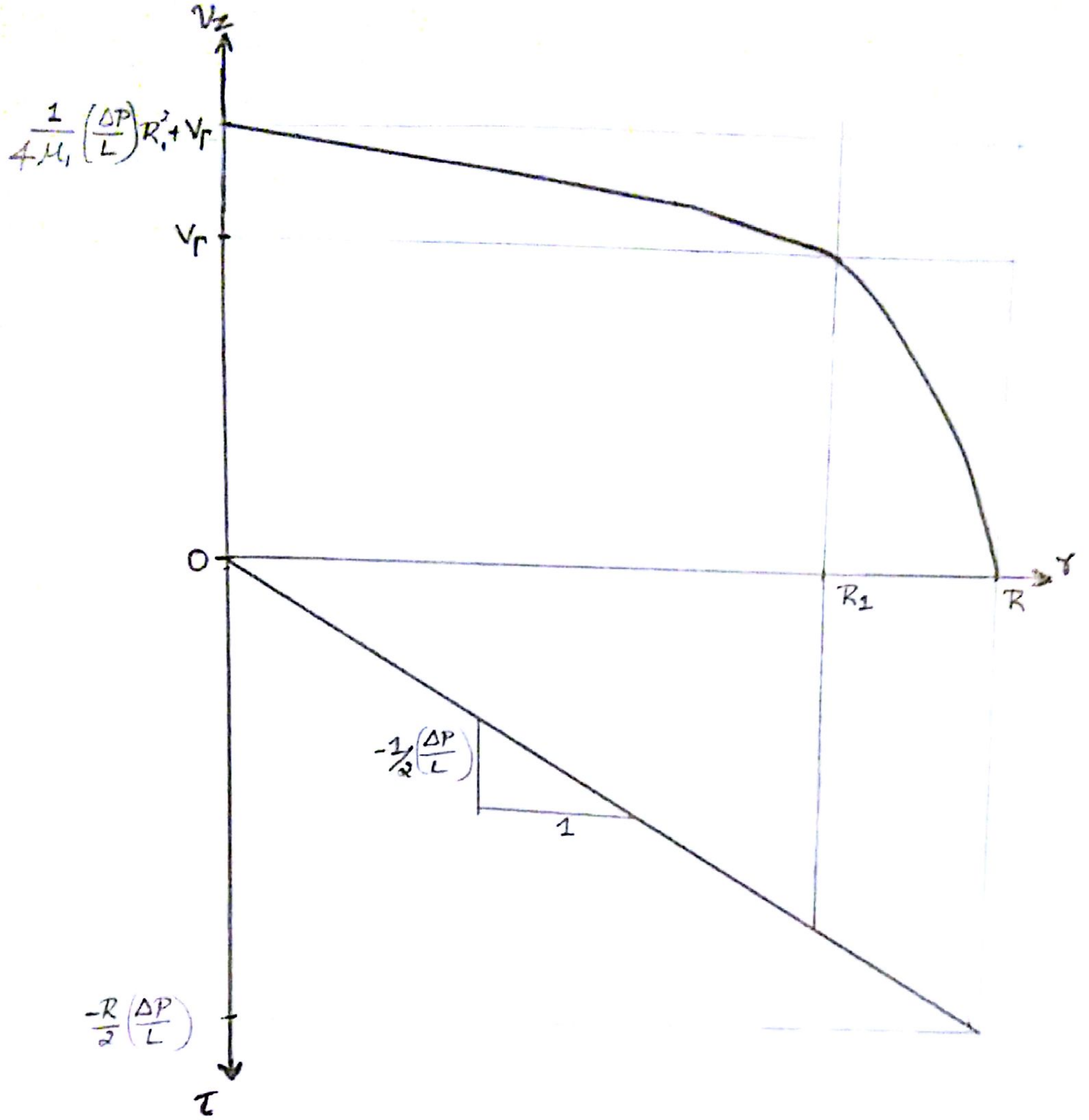

Figure ①:- velocity profile and shear stress profile across the entire pipe

Question (f):

The volumetric flow rate of oil Q_o is obtained by integrating velocity field in the core domain $0 \leq r \leq R_1$, i.e,

$$\boxed{Q_o = \int_0^{R_1} v_z(r)\, 2\pi r\, dr = \frac{\pi}{8\mu_1}\left(\frac{\Delta P}{L}\right)R_1^4 + \pi R_1^2 V_r},$$

and similarly, volumetric flow rate of water Q_w is obtained by integrating in shell domain $R_1 \leq r \leq R$:-

$$\boxed{Q_w = \int_{R_1}^{R} v_z(r)\, 2\pi r\, dr = \frac{\pi}{8\mu_2}\left(\frac{\Delta P}{L}\right)\left(R^2 - R_1^2\right)^2}$$

Problem ②:-

The solution presented here follows section 12.3 of 'CURRIE'S' book (Fundamental mechanics of Fluids) 3rd edition and also uses the solution provided in the lecture notes for infinite shock tube.

Since $(p_1 - p_0)/p_0 \ll 1$, the shock waves caused by breaking of diaphragm can be characterized as weak. In particular, they are isentropic.

As a consequence, the linearized form of the perturbed mass and momentum equation allow to establish two quantities, the Riemann invariants, that are constant along the so called characteristic lines:

$$\frac{u}{c} + \frac{1}{\gamma}\frac{p}{p_0} = \text{constant along } x - ct = \text{constant}$$

$$\frac{u}{c} + \frac{1}{\gamma}\frac{p}{p_0} = \text{constant along } x + ct = \text{constant}$$

(2)

Equations ② allow to evaluate the velocity and the pressure at any value of x and any value of 't', provided the solution is known for $t = 0$.

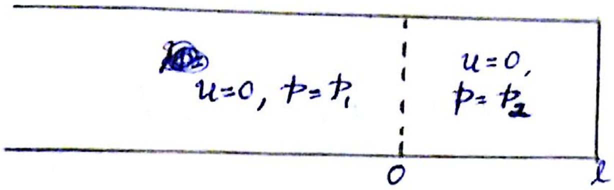

Figure ②:- Shock tube with a closed end & initial conditions.

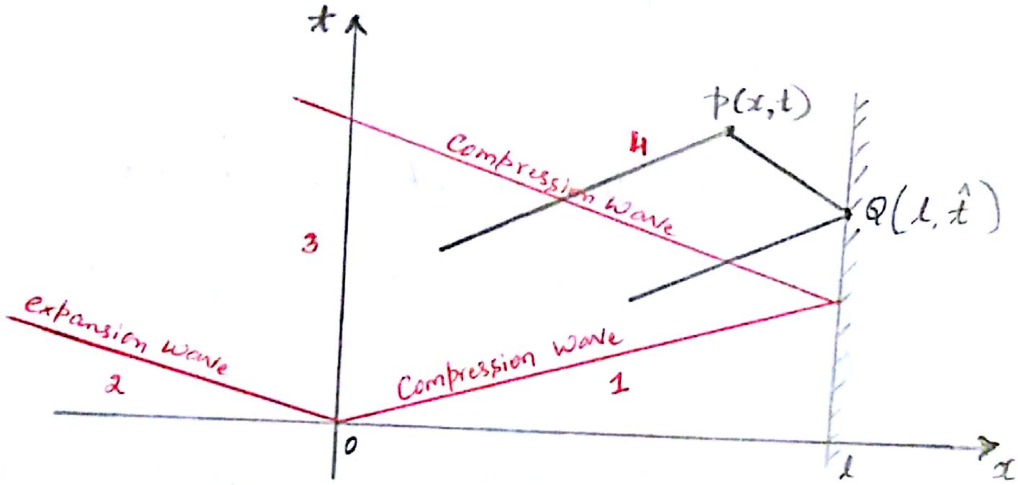

Figure ③:- x,t diagram showing the wave pattern that results from the breaking of diaphragm.

Figure ② represents the shock tube with closed end at $x=l$ and initial conditions. When the diaphragm bursts at $t=0$, the outgoing waves divide the x,t diagram into three distinct regions, in a similar way to the infinite shock tube problem:

* Region ① : that has not yet been affected by compression wave
* Region ② : that has not yet been affected by the expansion wave.
* Region ③ : that has been influenced by both outgoing waves.

However, in this case, when the compression waves impinges upon the closed end of tube, it is reflected as another wave that also travels at speed of sound, since infinitesimal perturbations are considered.

Thus behind the reflected wave a fourth region appears.

Figure ③ represents x, t diagram describing wave pattern which results from breaking of diaphragm.

In order to obtain the expressions for the velocity and the pressure at Region ④, an arbitrary point $P(x, t)$ is chosen, as seen in figure ③. The left characteristic line comes from region ③ and straight forwardly yields first equation:

$$\frac{u}{c} + \frac{1}{\gamma}\frac{p}{p_0} = \frac{1}{\gamma}\frac{p_1}{p_0}$$

Whereas the right characteristic line runs parallel to the reflected wave and from a point $Q(l, \hat{t})$ at the closed end. It follows from the fact that point Q lies on the solid boundary, that $u_Q = 0$.

Consequently, the only unknown at point 'Q' is the pressure. By drawing a characteristic line from 'Q' to Region ③, this quantity can be evaluated:-

$$\frac{1}{\gamma}\frac{p_Q}{p_0} = \frac{1}{\gamma}\frac{p_1}{p_0} \Rightarrow p_Q = p_1$$

In this manner, the second equation associated to the right characteristic line reads:-

$$\frac{u}{c} - \frac{1}{\gamma}\frac{p}{p_0} = -\frac{1}{\gamma}\frac{p_1}{p_0}$$

The solution for the resulting system of equations is $u = 0$ and $p = p_1$. Note that since pressure at region 4 is p_1 and pressure at region 3 is $(p_1 + p_0)/2 < p_1$ (recalling the solution at course notes), the reflected wave is also a compression wave.