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HOME WORK-1

① Let $\underline{F}$ & $\underline{G}$ be vector functions. Prove the following identities.

ⓐ $\nabla \cdot (\nabla \times \underline{F}) = 0$

Solution :-

W.K.T from LHS

$$\nabla \times \underline{F} = \varepsilon_{ijk} F_{k,j}$$

$$\nabla \cdot (\nabla \times \underline{F}) = \left[\varepsilon_{ijk} F_{k,j}\right]_{,i}$$

$$= \varepsilon_{ijk} F_{k,ij}$$

ε_{ijk}			
even permutation of ijk	123, 231, 312	$\rightarrow$	1
odd permutation of ijk	132, 213, 321	$\rightarrow$	-1

$$= \varepsilon_{123} F_{3,12} + \varepsilon_{231} F_{1,23} + \varepsilon_{312} F_{2,31} + \varepsilon_{132} F_{2,13} + \varepsilon_{213} F_{3,21} + \varepsilon_{321} F_{1,32}$$

$$= (1) F_{3,12} + (1) F_{1,23} + (1) F_{2,31} + (-1) F_{2,13} + (-1) F_{3,21} + (-1) F_{1,32}$$

$$= \cancel{F_{3,12}} + \cancel{F_{1,23}} + \cancel{F_{2,31}} - \cancel{F_{2,13}} - \cancel{F_{3,21}} - \cancel{F_{1,32}}$$

$$= \underline{\underline{0}} \rightarrow \text{RHS}$$

ⓒ $\nabla \cdot (\underline{F} \times \underline{G}) = \underline{G} \cdot \nabla \times \underline{F} - \underline{F} \cdot \nabla \times \underline{G}$

Solution :-

from LHS.

$$\underline{F} \times \underline{G} = \varepsilon_{ijk} F_j G_k$$

$$\nabla \cdot (\underline{F} \times \underline{G}) = \left[\varepsilon_{ijk} F_j G_k\right]_{,i}$$

$$= \varepsilon_{ijk} F_{j,i} G_k + \varepsilon_{ijk} F_j G_{k,i}$$

$= G_k \, \varepsilon_{ijk} F_{j,i} + \varepsilon_{ijk} G_{k,i} F_j$ $\qquad \varepsilon_{jik}$ = odd permutation of $ijk = -1$

$= G_k \, \varepsilon_{ijk} F_{j,i} - \varepsilon_{jik} G_{k,i} F_j$

$= \underline{G} \cdot \nabla \times \underline{F} - \underline{F} \cdot \nabla \times \underline{G} \rightarrow$ RHS

(b) $\nabla \times (\nabla \times \bar{F}) = \nabla(\nabla \cdot \bar{F}) - \nabla^2 \bar{F}$

Solution:-

From RHS:

$\bar{F} = (F_1, F_2, F_3)^T$

$$\nabla \times \bar{F} = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= e_1 \underbrace{\left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3} \right)}_{(u)} + e_2 \underbrace{\left(\frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1} \right)}_{(v)} + e_3 \underbrace{\left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right)}_{(w)}$$

$$\nabla \times (\nabla \times \bar{F}) = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ u & v & w \end{vmatrix}$$

$$= e_1 \left[\frac{\partial w}{\partial x_1} - \frac{\partial}{\partial x_3} v \right] + e_2 \left[\frac{\partial u}{\partial x_3} - \frac{\partial w}{\partial x_1} \right] + e_3 \left[\frac{\partial v}{\partial x_1} - \frac{\partial u}{\partial x_2} \right]$$

$$= \left\{ \left(\left[\frac{\partial^2 F_2}{\partial x_1 \partial x_2} - \frac{\partial^2 F_1}{\partial x_2^2} \right] - \left[\frac{\partial^2 F_1}{\partial x_3^2} - \frac{\partial^2 F_3}{\partial x_1 \partial x_3} \right] \right) + \left(\left[\frac{\partial^2 F_3}{\partial x_3 \partial x_2} - \frac{\partial^2 F_2}{\partial x_3^2} \right] - \left(\frac{\partial^2 F_2}{\partial x_1^2} - \frac{\partial^2 F_1}{\partial x_1 \partial x_2} \right] \right) + \left(\left[\frac{\partial^2 F_1}{\partial x_1 \partial x_3} - \frac{\partial^2 F_3}{\partial x_1^2} \right] - \left[\frac{\partial^2 F_3}{\partial x_2^2} - \frac{\partial^2 F_2}{\partial x_3 \partial x_2} \right] \right) \right\}$$

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(RHS)

$$\nabla\times(\nabla\times\bar{F}) = \frac{\partial^2 F_2}{\partial x_1 \partial x_2} - \frac{\partial^2 F_1}{\partial x_2^2} - \frac{\partial^2 F_1}{\partial x_3^2} + \frac{\partial^2 F_3}{\partial x_1 \partial x_3} + \frac{\partial^2 F_3}{\partial x_3 \partial x_2} - \frac{\partial^2 F_2}{\partial x_3^2}$$

$$- \frac{\partial^2 F_2}{\partial x_1^2} + \frac{\partial^2 F_1}{\partial x_1 \partial x_2} + \frac{\partial^2 F_1}{\partial x_1 \partial x_3} - \frac{\partial^2 F_3}{\partial x_1^2} - \frac{\partial^2 F_3}{\partial x_2^2} + \frac{\partial^2 F_2}{\partial x_2 \partial x_3}$$

Now Take RHS

$$\nabla(\nabla\cdot\bar{F}) - \nabla^2\bar{F}$$

$$\nabla(\nabla\cdot\bar{F}) = \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}\right)\left[\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}\right]$$

$$= \frac{\partial^2 F_1}{\partial x_1^2} + \frac{\partial^2 F_2}{\partial x_1 \partial x_2} + \frac{\partial^2 F_3}{\partial x_1 \partial x_3} + \frac{\partial^2 F_1}{\partial x_2 \partial x_1} + \frac{\partial^2 F_2}{\partial x_2^2}$$

$$+ \frac{\partial^2 F_3}{\partial x_2 \partial x_3} + \frac{\partial^2 F_1}{\partial x_3 \partial x_1} + \frac{\partial^2 F_2}{\partial x_3 \partial x_2} + \frac{\partial^2 F_3}{\partial x_3^2}$$

$$\nabla^2\bar{F} = \nabla^2 F_1 + \nabla^2 F_2 + \nabla^2 F_3$$

$$= \left(\frac{\partial^2 F_1}{\partial x_1^2} + \frac{\partial^2 F_1}{\partial x_2^2} + \frac{\partial^2 F_1}{\partial x_3^2}\right) + \left(\frac{\partial^2 F_2}{\partial x_1^2} + \frac{\partial^2 F_2}{\partial x_2^2} + \frac{\partial^2 F_2}{\partial x_3^2}\right)$$

$$+ \left(\frac{\partial^2 F_3}{\partial x_1^2} + \frac{\partial^2 F_3}{\partial x_2^2} + \frac{\partial^2 F_3}{\partial x_3^2}\right)$$

$$\therefore \left[\nabla(\nabla\cdot\bar{F}) - \nabla^2\bar{F}\right] = \cancel{\frac{\partial^2 F_1}{\partial x_1^2}} + \frac{\partial^2 F_2}{\partial x_1 \partial x_2} + \frac{\partial^2 F_3}{\partial x_1 \partial x_3} + \frac{\partial^2 F_1}{\partial x_2 \partial x_1} + \cancel{\frac{\partial^2 F_2}{\partial x_2^2}}$$

$$+ \frac{\partial^2 F_3}{\partial x_2 \partial x_3} + \frac{\partial^2 F_1}{\partial x_3 \partial x_1} + \frac{\partial^2 F_2}{\partial x_3 \partial x_2} + \cancel{\frac{\partial^2 F_3}{\partial x_3^2}} - \cancel{\frac{\partial^2 F_1}{\partial x_1^2}} -$$

$$\frac{\partial^2 F_1}{\partial x_2^2} - \frac{\partial^2 F_1}{\partial x_3^2} - \frac{\partial^2 F_2}{\partial x_1^2} - \cancel{\frac{\partial^2 F_2}{\partial x_2^2}} - \frac{\partial^2 F_2}{\partial x_3^2} - \frac{\partial^2 F_3}{\partial x_1^2}$$

$$- \frac{\partial^2 F_3}{\partial x_2^2} - \cancel{\frac{\partial^2 F_3}{\partial x_3^2}}$$

$$= \frac{\partial^2 F_2}{\partial x_1 \partial x_2} - \frac{\partial^2 F_1}{\partial x_2^2} - \frac{\partial^2 F_1}{\partial x_3^2} + \frac{\partial^2 F_3}{\partial x_1 \partial x_3} + \frac{\partial^2 F_3}{\partial x_3 \partial x_2} - \frac{\partial^2 F_2}{\partial x_3^2}$$

$$- \frac{\partial^2 F_2}{\partial x_1^2} + \frac{\partial^2 F_1}{\partial x_1 \partial x_2} + \frac{\partial^2 F_1}{\partial x_1 \partial x_3} - \frac{\partial^2 F_3}{\partial x_1^2} - \frac{\partial^2 F_3}{\partial x_2^2} + \frac{\partial^2 F_2}{\partial x_2 \partial x_3}$$

$$RHS = (RHS)$$

or

$$\nabla \times (\nabla \times \underline{F}) = \left[\varepsilon_{ijk} \left(\varepsilon_{klm} F_{m,l}\right)\right]_{,j}$$

$$= \varepsilon_{ijk} \varepsilon_{lmk} F_{m,lj}$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) F_{m,lj}$$

$$= \delta_{il}\delta_{jm} F_{m,lj} - \delta_{im}\delta_{jl} F_{m,lj}$$

$$= \delta_{il} F_{j,lj} - \delta_{jl} F_{i,lj}$$

$$= F_{j,ij} - F_{i,jj}$$

$$\nabla \times (\nabla \times \underline{F}) = \nabla(\nabla \cdot \underline{F}) - \nabla^2 \underline{F}$$

Q2. The integral form of the second law of thermodynamics reads,

$$\frac{D}{Dt}\int_{V_t} \rho s \, dV \geq -\int_{S_t} \frac{\bar{q}\cdot\bar{n}}{T} \, dS$$

where s is entropy/unit mass.

under following assumptions:

- Newtonian fluid with bulk viscosity ($K \geq 0, \mu > 0$)
- Fourier's law for heat conduction ($\bar{q} = -K \nabla T, K > 0$)

Show that above inequality always holds.

Solution:

From 1st law of Thermodynamics

$$dQ = de + dW$$

$$dQ = de + P\,dV \quad -(i)$$

$dW = P\,dV$

$de \rightarrow$ Internal energy

From 2nd law of Thermodynamics

$$dQ = T\,ds \quad -(ii)$$

from (i) & (ii)

$$T\,ds = de + P\,dv \quad (\text{for all infinitesimal changes})$$

$v = 1/\rho$

$dv = \frac{-1}{\rho^2} d\rho$

$$T\,ds = de - \frac{P}{\rho^2} d\rho$$

$$T\frac{Ds}{Dt} = \frac{De}{Dt} - \frac{P}{\rho^2}\frac{D\rho}{Dt}$$

$$\rho T\frac{Ds}{Dt} = \rho\frac{De}{Dt} - \frac{P}{\rho}\frac{D\rho}{Dt} \quad -(iii)$$

W.K.T

$$\rho\frac{De}{Dt} = \sigma : \nabla v - \nabla\cdot q$$

$$\rho\frac{De}{Dt} = -p\,\nabla\cdot v - \nabla\cdot q + \Phi$$

where Φ = Dissipation energy,

$$\Phi = \lambda(\nabla\cdot v)^2 + 2\mu\,\nabla^2 v : \nabla v$$

& Φ is positive, since $\lambda(\nabla\cdot v)^2 + 2\mu\nabla^2 v : \nabla v$ is positive

Sub in (iii)

$$\therefore\ \rho T\frac{Ds}{Dt} = \left(-p\,\nabla\cdot v - \nabla\cdot q + \Phi\right) - \frac{P}{\rho}\frac{D\rho}{Dt}$$

$$= -\frac{P}{\rho}\Big(\underbrace{\frac{D\rho}{Dt} + \rho\,\nabla\cdot v}_{\text{(Mass Continuity Eqn)}}\Big)^{\nearrow 0} - \nabla\cdot q + \Phi$$

$$\rho T\frac{Ds}{Dt} = -\nabla\cdot q + \Phi$$

$$\rho \frac{Ds}{Dt} = -\frac{\nabla \cdot q}{T} + \frac{\Phi}{T}$$

$$\nabla \cdot \left(\frac{q}{T}\right) = \frac{1}{T} \nabla \cdot q - \frac{q \cdot \nabla T}{T^2}$$

or

$$-\frac{\nabla \cdot q}{T} = -\nabla \cdot \left(\frac{q}{T}\right) - \frac{q \cdot \nabla T}{T^2}$$

$$\therefore \rho \frac{Ds}{Dt} = -\nabla \cdot \left(\frac{q}{T}\right) - \frac{q \, \nabla T}{T^2} + \frac{\Phi}{T}$$

$$\rho \frac{Ds}{Dt} = -\nabla \cdot \left(\frac{q}{T}\right) + \underbrace{\left(\frac{K \nabla T \cdot \nabla T}{T^2} + \frac{\Phi}{T}\right)}_{\text{Positive term}} \quad (q = -K \nabla T)$$

$$\int_{V_t} \rho \frac{Ds}{Dt} \, dV = \int_{V_t} -\nabla \cdot \left(\frac{q}{T}\right) dV + \int_{V_t} \left(\frac{K \nabla T \cdot \nabla T}{T^2} + \frac{\Phi}{T}\right) dV$$

using Reynold's Lemma Concept

$$\frac{D}{Dt} \int \rho s \, dV = \int_{V_t} -\nabla \cdot \left(\frac{q}{T}\right) dV + \int_{V_t} \left(\frac{K \nabla T \cdot \nabla T}{T^2} + \frac{\Phi}{T}\right) dV$$

using divergence Theorem

$$\frac{D}{Dt} \int_{V_t} \rho s \, dV = -\int_{S_t} \frac{\bar{q} \cdot \bar{n} \, dS}{T} + \underbrace{\int_{V_t} \left(\frac{K \nabla T \cdot \nabla T}{T^2} + \frac{\Phi}{T}\right) dV}_{\text{This is always positive}}$$

$$\boxed{\therefore \frac{D}{Dt} \int_{V_t} \rho s \, dV \geq -\int_{S_t} \frac{\bar{q} \cdot \bar{n} \, dS}{T}}$$

$\therefore$ Inequality always holds