① $f = (v_j, a, \rho_s \cdot \rho_c, \mu_e, g)$

	v_j	a	ρ	μ	g
M	0	0	1	1	0
L	-1	1	-3	-1	1
T	-1	0	0	-1	-2

$n - K = 2$

$f(\Pi_1, \Pi_2) = 0$

$\Pi_1 = v_j\, a^a \rho^b g^c$

$\Pi_2 = \mu\, a^a \rho^b g^c$

$$M^0 L^0 T^0 = (LT^{-1})(L)^a (ML^{-3})^b (LT^{-2})^c = M^b L^{1+a-3b+c} T^{-1-2c}$$

$0 = b$

$0 = 1 + a - \cancel{3b} + c \Rightarrow a = -1 - c$

$0 = -1 - 2c \Rightarrow c = -\frac{1}{2}$

$a = -\frac{1}{2}$

$$\left[\Pi_1 = \frac{v_j}{\sqrt{a \cdot c}}\right]$$

② a) $\frac{\partial}{\partial t} \int_V f(x,t)\, dV = \int_V \frac{\partial f(x,t)}{\partial t} \partial V + \int_S f(x,t)\, v \cdot n\, dS$

$$\frac{D}{Dt} \int_V \rho s\, dV \geq - \int_S \frac{q \cdot n}{T} dS$$

$$\frac{D}{Dt} \int_V \rho s\, dV = \int_V \frac{\partial \rho s}{\partial t} \partial V + \int_S (\rho s \otimes s) \cdot n\, dS = \int_V \left(\frac{\partial \rho s}{\partial t} + \nabla \cdot (\rho s \otimes s) \right) \partial V$$

$[s \otimes s]_{ij} = v_i v_j \quad \frac{D}{Dt} \int_V \rho s\, dV = \int_V \rho \frac{Ds}{Dt} \partial V$

$$- \int_S \frac{q}{T} \cdot n\, dS = - \int_V \nabla \cdot \frac{q}{T} \partial V$$

$$\frac{D}{Dt} \int_V \rho s\, dV \geq \int_S \frac{q \cdot n}{T} dS \Rightarrow \int_V \rho \frac{Ds}{Dt} \partial V \geq - \int_V \nabla \cdot \frac{q}{T} \partial V$$

$$\rho \frac{Ds}{Dt} + \nabla \cdot \frac{q}{T} \geq 0$$

b) $$T\,ds = de + p\,d\left(\frac{1}{\rho}\right) \Rightarrow T\,ds = \partial e - p\frac{\partial\rho}{\rho^2} \Rightarrow ds = \frac{1}{T}\left(\partial e - \frac{P\,\partial\rho}{\rho^2}\right)$$

$$\rho\frac{Ds}{Dt} + \nabla\cdot\frac{q}{T} \geq 0 \qquad s = \frac{1}{T}\left[\int_0^e de - P\int_0^\rho \frac{\partial\rho}{\rho^2}\right] = \frac{1}{T}\left(e + P\frac{1}{\rho}\right)$$

$$\rho\frac{Ds}{Dt} = \left(\frac{\partial s}{\partial t} + \nabla(s v)\right)\rho = \left(\frac{\partial s}{\partial t} + s\cdot\nabla v + v\cdot\nabla s\right)\rho \Rightarrow v\cdot\nabla s = \left(-\frac{P}{\rho^2 T}\right)\nabla\cdot v$$

$$\rho\left[\frac{\partial s}{\partial t} + s\cdot\nabla v + v\cdot\nabla s\right] + \nabla\cdot\frac{q}{T} \geq 0 \Rightarrow \rho\left[\frac{1}{\partial t}\left(\frac{1}{T}\left(\partial e - \frac{P\,\partial\rho}{\rho^2}\right)\right) + s\cdot\nabla v + v\cdot\nabla s\right] + \nabla\cdot\frac{q}{T} \geq 0$$

$$\Rightarrow \rho\left[\left(\frac{1}{\partial t}\left(\frac{1}{T}\left(\partial e - \frac{P\,\partial\rho}{\rho^2}\right)\right)\right) + \frac{1}{T}\left(e+\frac{P}{\rho}\right)\nabla\cdot v + \nabla\cdot v\left(-\frac{P}{\rho^2 T}\right)\right] + \nabla\cdot\frac{q}{T} \geq 0$$

$$\Rightarrow \frac{1}{T}\left(\frac{\rho\,\partial e}{\partial t} - \frac{P\rho\,\partial\rho}{\rho^2\,\partial t}\right) + \frac{\rho}{T}v\cdot\nabla e + \frac{\rho}{T\rho}P\nabla\cdot v - \frac{P\rho\nabla\cdot v}{\rho^2 T} + \nabla\cdot\frac{q}{T} \geq 0$$

$$\rho\frac{\partial e}{\partial t} - \frac{P}{\rho^2}\rho\frac{\partial e}{\partial t} + \rho v\cdot\nabla e + P\nabla\cdot v - \frac{P\rho}{\rho^2}\nabla\cdot v + T\left(\nabla\cdot\frac{q}{T}\right) \geq 0$$

$$\rho\frac{\partial e}{\partial t} - \frac{P}{\rho^2}\left(\rho\frac{\partial e}{\partial t} + \rho\nabla\cdot v\right) + \rho v\cdot\nabla e + T\left(\nabla\cdot\frac{q}{T}\right) + P\nabla\cdot v \geq 0$$

$$\underbrace{\rho\frac{\partial e}{\partial t} + \rho v\cdot\nabla e}_{\sigma:\nabla v - \nabla\cdot q} + T\left(\nabla\cdot\frac{q}{T}\right) + P\nabla\cdot v \geq 0 \Rightarrow \sigma:\nabla v - \nabla\cdot q + P\nabla\cdot v + T\left(\nabla\cdot\frac{q}{T}\right) \geq 0$$

$$\sigma:\nabla v - \nabla\cdot q + P\nabla\cdot v + \cancel{T}\left(\frac{\nabla\cdot q\,T - q\cdot\nabla T}{T^2}\right) \geq 0$$

$$\sigma:\nabla v - \nabla\cdot q + P\nabla\cdot v + \nabla\cdot\frac{q\cancel{T}}{\cancel{T}} - \frac{q\cdot\nabla T}{T} \geq 0$$

$$\sigma:\nabla v + P\nabla\cdot v - \frac{q\cdot\nabla T}{T} \geq 0$$

c) $\sigma : \nabla v + p \nabla \cdot v - \frac{q \cdot \nabla T}{T} \geq 0$ $\quad q = -K \nabla T \Rightarrow \frac{K (\nabla T)^2}{T} \geq 0$

$\sigma = -pI + \lambda (\nabla^s v) I + 2\mu \nabla^s v$

$-pI : \nabla v + p \nabla \cdot v + \lambda (\nabla^s v) I : \nabla v + 2\mu \nabla^s v : \nabla v + \frac{K (\nabla T)^2}{T} \geq 0$

$K = \lambda + \frac{2}{3}\mu \Rightarrow \begin{matrix} K \geq 0 \\ \mu > 0 \end{matrix} \qquad \lambda = K - \frac{2}{3}\mu$

$\cancel{-p \nabla \cdot v} + \cancel{p \nabla \cdot v} + \left[K (\nabla^s v) I + 2\mu \underbrace{\left(\nabla^s v - \frac{1}{3} (\nabla^s v) I \right)}_{} \right] : \nabla v \geq 0$

$K (\nabla^s v) I : \nabla v$

$k (\nabla v)^2 \geq 0$

$dev (\nabla^s v) : (\nabla v) \Rightarrow dev(\nabla^s v) : (\nabla^s v + \nabla^w v)$

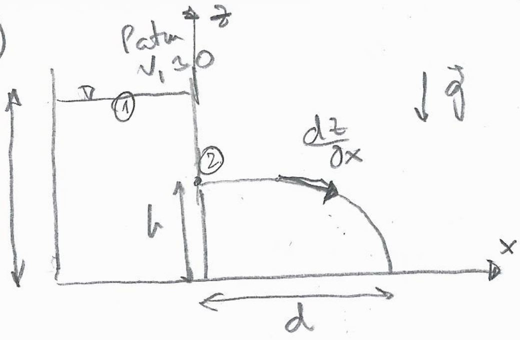
$dev(\nabla^s v) : \nabla^s v + dev(\nabla^s v) : \nabla^w v$

$dev(\nabla^s v) : dev(\nabla^s v) + \left[\left(\nabla^s v - \frac{1}{3} \nabla \cdot v I \right) : \nabla^w v \right]$

$dev(\nabla^s v) : dev(\nabla^s v) + \left[\left(\cancel{\underset{0}{\nabla^s v : \nabla^w v}} \right) - \frac{1}{3} \nabla \cdot v \cancel{(\underbrace{I : \nabla^w v}_{tr(\nabla^w v) = 0})} \right]$

$dev(\nabla^s v) : dev(\nabla^s v) \geq 0$

3



1 stationary

a) 1

$$\int_1^2 \frac{\partial v}{\partial t} dx + \frac{V_2^2}{2} + \cancel{\frac{P_2}{\rho}} + g z_2 = \cancel{\frac{P_1}{\rho}} + \cancel{\frac{V_1^2}{2}} + g z_1$$

$$\frac{V_2^2}{2} = g z_1 - g z_2 \Rightarrow \frac{V_2^2}{2} = g(H-h)$$

$$V_2 = \sqrt{2g(H-h)}$$

$$v = (u, w) = \left(\sqrt{2g(H-h)}, 0\right)$$

b)

$$\frac{dz}{\partial x} = \frac{w}{u} \qquad \frac{dz}{\partial x} = \frac{w}{u} \rightarrow \int_h^0 \frac{u}{w} \partial z = \int_0^d \partial x$$

$$\int_h^0 \frac{\sqrt{2g(H-h)}}{w} \partial z = d \Rightarrow \sqrt{2g(H-h)} \cdot \int_h^0 \frac{1}{w} \partial z = d$$

$$\sqrt{2g(H-h)} \cdot \int_h^0 \frac{1}{w} dz = d$$

Bernoulli between any point of the waterjet

$$\frac{V_2^2}{2} + \cancel{\frac{P}{\rho}} + gh = \frac{V^2}{2} + \cancel{\frac{P}{\rho}} + gz \Rightarrow \frac{V_2^2}{2} + g(h-z) = \frac{V^2}{2}$$

$$\frac{V_2^2}{2} + 2g(h-z) = \frac{V^2}{2} \Rightarrow V_2^2 + 2g(h-z) = V^2 = \left(\sqrt{w^2+u^2}\right)^2$$

$$2g(h-z) = w^2 \qquad w = \pm\sqrt{2g(h-z)} \qquad v = \left(\sqrt{2g(H-h)}, -\sqrt{2g(h-z)}\right)$$

$$\int_h^0 \frac{1}{-\sqrt{2g(h-z)}} dz = 2\sqrt{2g(h-z)}\Big|_h^0 = 2\sqrt{2gh}$$

$$2\sqrt{2g(H-h)}\sqrt{2gh} = d \Rightarrow \boxed{4g\sqrt{h(H-h)} = d}$$

c) $H = 10$

$$d = \sqrt{h(10-h)h}\; 4g \qquad d_{max} \rightarrow \frac{\partial(d)}{\partial h} = \frac{\partial}{\partial h}(10h - h^2)^{1/2} = \frac{1}{2}(10h-h^2)^{-1/2}(10-2h) = 0$$

$$0 = \frac{1}{2}\frac{10-2h}{\sqrt{10h-h^2}} \rightarrow 10 - 2h = 0 \Rightarrow \boxed{h = 5}$$