Universitat Politècnica de Catalunya
Numerical Methods in Engineering
Advanced Fluid Mechanics

Assignment 2

Navier Stokes equations
Boundary Layer

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1 Exercise 1

1.1 Problem

Consider the steady laminar flow through the annular space formed by two coaxial tubes. The radius of the outer tube is R_1 and the radius of the inner tube is R_2. The flow is along the axis of the tubes and maintained by a constant pressure gradient $\frac{\partial p}{\partial z}$, where the z direction is taken along the axis of the tubes. We are asked to

- Finding the velocity field
- Finding the point with the highest local velocity
- Finding the volumetric flowrate
- Defining $\phi = R_2/R_1$, to compare the flowrate for $\phi \to 0$ with the Hagen-Poseuille flow in a cylindrical pipe.

1.2 Solution

We'll start off with the hypotheses

1. The fluid is Newtonian.
2. The fliud is incompressible.
3. The flow is in a steady state (i.e $\partial/\partial t = 0$).
4. The flow is axially symmetrical (i.e $\partial/\partial\theta = 0$).
5. The velocity only goes in the Z direction: $\boldsymbol{V} = [0, 0, v_z]$
6. Gravity (and any other body force) is neglected.

The boundary conditions are typical for static walls:

$$\begin{aligned} \boldsymbol{V}(r = R_1) = [0, 0, 0] \\ \boldsymbol{V}(r = R_2) = [0, 0, 0] \end{aligned} \tag{1}$$

We'll start off with the continuity equation:

$$\frac{1}{r}\frac{\partial r u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$

Since both radial and angular velocity are zero (hypothesis 5), we obtain:

$$\frac{\partial u_z}{\partial z} = 0 \tag{2}$$

This together with hypothesis 4 implies that u_z only changes in the r direction: $u_z = f(r)$.
Here is the momentum balance equation for incompressible Newtonian fluids in the z-axis in cylindrical coordinates. The other two axes are not useful to us.

$$\rho\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \rho g_z + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z}\right] \tag{3}$$

From hypotheses 3 and 5 we banish the left hand side.

$$-\frac{\partial P}{\partial z} + \rho g_z + \mu \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z} \right] = 0$$

Hypothesis 4 and equation 2 allow removing the second and third terms in the brackets, respectively. The equation becomes:

$$-\frac{\partial P}{\partial z} + \rho g_z + \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r} \right) = 0$$

Hypothesis 6 allows one last simplification, removing the body forces. Rearranging, and expanding the double derivative we obtain our differential equation:

$$\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} - \frac{1}{\mu}\frac{\partial P}{\partial z} = 0 \tag{4}$$

We know that u_z only depends on r.We can rewrite it as the following ODE:

$$u_z''(r) + \frac{1}{r}u_z'(r) = \frac{1}{\mu}\frac{\partial P}{\partial z} \tag{5}$$

The solution to this ordinary differential equation is

$$u_z(r) = \frac{1}{4\mu}\frac{\partial P}{\partial z}r^2 + A\,ln(r) + B \tag{6}$$

We have two integration constants and two boundary conditions, hence obtaining the following 2×2 equation system:

$$\left.\begin{aligned} 0 &= \frac{1}{4\mu}\frac{\partial P}{\partial z}R_1^2 + A\,ln(R_1) + B \\ 0 &= \frac{1}{4\mu}\frac{\partial P}{\partial z}R_2^2 + A\,ln(R_2) + B \end{aligned}\right\} \tag{7}$$

After substantial algebraic manipulation we obtain the solution:

$$\boxed{u_z(r) = -\frac{R_1^2 - R_2^2}{4\mu}\frac{\partial P}{\partial z}\left(\frac{r^2 - R_2^2}{R_1^2 - R_2^2} - \frac{ln(r/R_2)}{ln(R_1/R_2)} \right)} \tag{8}$$

If we want to obtain the position with highest local velocity, we must differentiate.

$$\frac{dv_z}{dr} = 0 \tag{9}$$

After some manipulation:

$$\boxed{r\Big|_{v\text{ max}} = \sqrt{\frac{R_1^2 - R_2^2}{ln(R_1^2/R_2^2)}}} \tag{10}$$

In order to obtain the volumetric flowrate we must integrate along a surface normal to the flow, that is, an annularsection of the pipe:

$$Q = 2\pi \int_{R_2}^{R_1} u_z(r)\, r\, dr \tag{11}$$

If we substitute $\phi = R_2/R_1$, we obtain

$$\boxed{Q = \frac{\pi R_1^4}{8\mu}\frac{\partial P}{\partial z}\left[(\phi^4 - 1) - \frac{(\phi^2 - 1)^2}{\ln(\phi)}\right]} \tag{12}$$

Finally, if we take the limit:

$$\boxed{\lim_{\phi \to 0} Q = -\frac{\pi R_1^4}{8\mu}\frac{\partial P}{\partial z}} \tag{13}$$

we obtain the same flow than for Hagen-Poseuille flow in a cilindrical pipe.

$$Q_{\text{HP}} = -\frac{\pi R^4}{8\mu}\frac{\partial P}{\partial z} \tag{14}$$

2 Exercise 2

2.1 Problem

Use the Kármán-Pohlhausen approximation to compute the boundary layer solution for an uniform flow over a flat plate. Assume a quadratic polynomial form for the velocity profile:

$$\frac{u}{U} = a + b\frac{y}{\delta} + c(\frac{y}{\delta})^2 \tag{15}$$

and use the following boundary conditions:

$$u = 0 \qquad\qquad at \quad y = 0 \tag{16}$$

$$u = U, \quad \frac{\partial u}{\partial y} = 0 \qquad at \quad y = \delta \tag{17}$$

Compare this solution with Blasius exact solution.

2.2 Solution

From the boundary condition at $y = 0$ we get:

$$a = 0 \tag{18}$$

Then, the expression becomes:

$$\frac{u}{U} = b\frac{y}{\delta} + c(\frac{y}{\delta})^2 \tag{19}$$

$$\frac{\partial u}{\partial y} = b\frac{U}{\delta} + 2c\frac{U}{\delta^2}y \tag{20}$$

And from the boundary conditions at $y = \delta$ we get:

$$b = 2 \qquad c = -1 \tag{21}$$

Therefore, we get the expression:

$$\frac{u}{U} = 2\frac{y}{\delta} - (\frac{y}{\delta})^2 \tag{22}$$

After defining the constant values we may calculate the momentum thickness as:

$$U^2\theta = \int_0^\delta u(U-u)dy = \frac{2}{15}U^2\delta \tag{23}$$

$$\theta = \frac{2}{15}\delta \tag{24}$$

Substituting (23) in the momentum integral equation:

$$\frac{d(U^2\theta)}{dx} = \frac{2}{15}U^2\frac{d\delta}{dx} = \frac{\tau_0}{\rho} \tag{25}$$

Where τ_0 is the shear stress on the plate surface and can also be expressed as:

$$\frac{\tau_0}{\rho} = \upsilon(\frac{\partial u}{\partial y})_0 = 2\upsilon\frac{U}{\delta} \tag{26}$$

Now, making an equality between (25) and (26) we have:

$$\frac{d\delta}{dx} = 15\frac{\upsilon}{U\delta} \tag{27}$$

Integrating:

$$\frac{\delta^2}{2} = 15\frac{\upsilon x}{U} + C1 \tag{28}$$

Since $\delta(0) = 0$, we know that $C1 = 0$ and the expression becomes:

$$\frac{\delta}{x} = \sqrt{\frac{30}{Re}} \tag{29}$$

Where Re is the Reynolds number. For the Kármán-Pohlhausen approximation we may also compute:

$$\frac{\theta}{x} = \sqrt{\frac{8}{15Re}} \tag{30}$$

Summarizing, we have the following results for the approximation:

$$\boxed{\frac{\delta}{x} = \sqrt{\frac{30}{Re}} \qquad \frac{\theta}{x} = \sqrt{\frac{8}{15Re}}} \tag{31}$$

We must now compare these results with the Blasius exact solution, which defines the following expressions:

$$\frac{\delta}{x} = \sqrt{\frac{25}{Re}} \qquad \frac{\theta}{x} = \frac{0.664}{\sqrt{Re}} \tag{32}$$

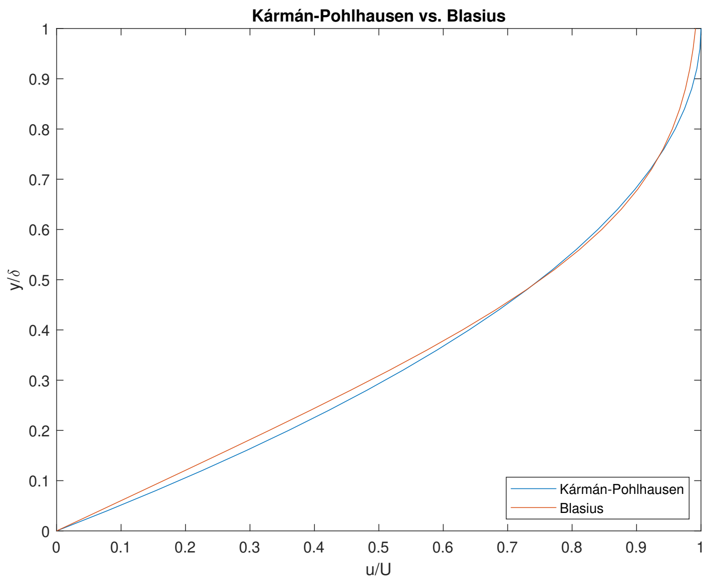

Figure 1: Comparison between Kármán-Pohlhausen and Blasius [1] velocity profiles.

Dividing the expressions found for the approximation by their respective equivalents in the exact solution, we have:

$$\boxed{\frac{\delta_k}{\delta_b} = \sqrt{\frac{30}{25}} = 1.0954 \qquad \frac{\theta_k}{\theta_x} = \frac{0.7303}{0.664} = 1.0998} \tag{33}$$

Where δ_k and θ_k correspond to the approximation, while δ_b and θ_b correspond to the exact solution. For both expressions we have a difference of approximately 10%.

References

[1] Frank M. White. *Fluid Mechanics*. 7th. McGraw-Hill, 2001. ISBN: 978-0-07-352934-9.