ADVANCED FLUID MECHANICS
Master of Science in Computational Mechanics/ Numerical Methods
Fall Semester 2015
Homework 2: Bernoulli's equation and Irrotational flow
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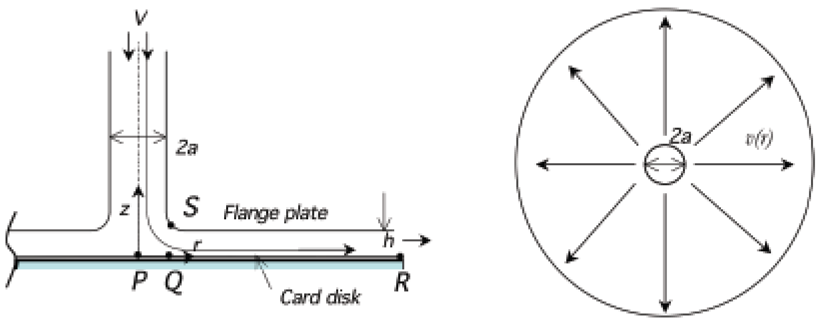
Exercise 1· Compressed air is blown axially down the narrow tube (of radius a) and flows radially outwards below a flange plate (a flat circular disk of radius *R* >> *a*).This flange is separated from a circular cardboard disk by a distance *h* < *a*. As the flow rate is increased, eventually the net pressure force pushing down on the cardboard disk becomes negative and lifts the disk upwards.

The flow in this device is complex, especially in the region where the flow rearranges from axial downflow to radial outflow. However, we shall model the streamlines in the flow as shown in the figure 1 below: The average inlet velocity is *V* and the static pressure is p_0. The point



P is a stagnation point. To find the net pressure force on the disk we need to evaluate the pressure distribution $p(r)$ along the streamline PQR. Model the flow along PQR as inviscid with streamlines that are purely radial with $v_r = f(r)$

a) In the region PQ the air must accelerate radially from zero at P in order to accommodate the additional air arriving from the axial inflow in the tub above. We thus propose that $v_r = C_1 r$ for *r* < *a*. Show that for any arbitrarily radial position *r* > *a* the velocity field is of the form:

$$v_r = \frac{C_2}{R} \, for \, r > a$$

And evaluate the constants in terms of the parameters *h, a, V.*

It is known the velocity in the inlet as well as the area. On the other hand we will consider the area of the outlet of a cylindrical volume in the flange of $\text{lateral area} = 2\pi r h\ (r > a)$, being the velocity outwards of this surface v_r so that we can apply the principle of mass conservation as follows:

$$V\pi a^2 = 2\pi r h v_r$$

$$v_r = \frac{a^2 V}{2hr}$$

$$\boxed{C_2 = \frac{a^2 V}{2h}}$$

For the C_1 constant we know that at $r = a$ both velocities have to be the same:

$$\frac{c_2}{a} = C_1 a$$

$$\boxed{C_1 = \frac{V}{2h}}$$

b) Using your velocity field, evaluate the gage pressure field $p(r) - p_o$ along the streamline $\overline{PQR}$ and plot your answer graphically, crealy indicating the shapes of the curve draw and labeling any points of maximum or minimum gage pressure.

Taking into account that the flow is inviscid, incompressible and the body forces are conservative we can solve the problem applying the Bernoulli's principle along a stream line going from the point 0 to the point P. It is known the static pressure in the inlet as well as the velocity V, moreover the velocity in the stagnation point is obviously 0 so that $v_p = 0$. Thus we get:

$$\frac{1}{2}V^2 + \frac{P_0}{\rho} + g z_o - \left(\frac{1}{2}{v_p}^2 + \frac{P_p}{\rho} + g z_p\right) = 0$$

Considering that $v_p = 0$ and the energy due to height negligible we obtain:

$$P_p = P_0 + \frac{1}{2}\rho V^2 \qquad (1)$$

Now, in order to obtain different expressions for the gage pressure depending on the considered zone, we have to apply Bernoulli between P and P' (P' a point located between P and Q where $0 < r < a$):

$$\frac{1}{2}v(r)^2 + \frac{P(r)}{\rho} + gz - \left(\frac{1}{2}{v_p}^2 + \frac{P_p}{\rho} + gz\right) = 0$$

In this case we can directly cancel the height terms so they are equal to each other, again $v_p = 0$, so:

$$P(r) = P_p - \frac{1}{2}\rho v(r)^2$$

But in this zone we know data so:

$$v_r = C_1 r$$

Then substituting this expression we get:

$$P(r) = P_p - \frac{1}{2}\rho\left(\frac{V}{2h}r\right)^2 \tag{2}$$

The same way we can apply Bernoulli between Q and Q' (Q' a point located between points Q and R where *a* < *r* < *R* obtaining exactly the same expression (2) but in this case the expression of the velocity in this zone is different so that, then proceeding the same way as above:

$$P(r) = P_p - \frac{1}{2}\rho\left(\frac{a^2V}{2hr}\right)^2 \tag{3}$$

Finally we obtain the expressions of $p(r) - p_o$ along the streamline $\overline{PQR}$ by substituting (1) into (2) and (3):

$$p(r) - p_o = \begin{cases} \frac{1}{2}\rho V^2 - \rho\frac{V^2}{8h^2}r^2 & (0 < r < a) \\ \frac{1}{2}\rho V^2 - \rho\frac{a^4V^2}{8h^2r^2} & (a < r < R) \end{cases}$$

To plot the solution we have taken advantage of using the free software Octave. We have defined the parameters implied in the problem in such a way that the pressure values are either positive or negative in some points of the cardboard. So we select:

$$R = 100; a = 10; h = 1.5; d = 1.2; V = 0.25$$

Plotting the radial distance vs the gage pressure we get:

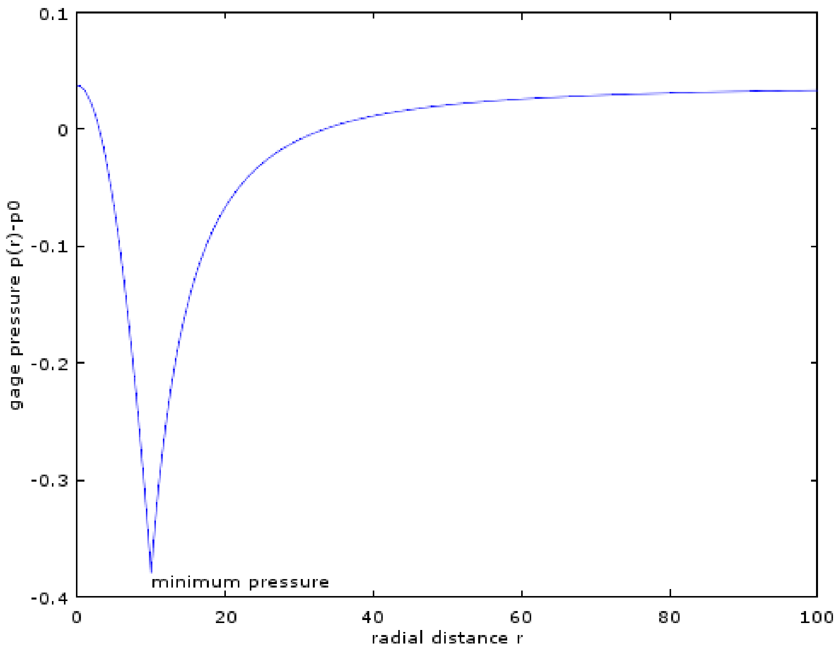


Figure 1. Plot of the pressure profile $p(r) - p_0$.

Taking the minimum pressure value $P_{min} = \frac{1}{2}\rho V^2 - \rho\frac{V^2}{8\,h^2}a^2$ at $r = a$, and maximum pressure $P_{max} = \frac{1}{2}\rho V^2$ at the stagnation point $r = 0$.

c) Integrate your answer for $p(r)$ over the entire disk to show the total downward pressure acting on the disk is:

$$F_p = \frac{1}{2}\pi V^2\left[R^2 - \frac{a^4}{8h^2}\right] + \pi\rho V^2\left(\frac{a^4}{4h^2}\right) ln\left(\frac{a}{R}\right) \quad (4)$$

Explain why this force can be negative, and find the inflow air velocity V required to levitate a 10 gram cardboard disk when a=1cm, R = 5cm, h=0.1cm.

In order to integrate $p(r) - p_o$ all over the entire cardboard we should account for separating the integral in two, the first one because we have two expressions, one for $0 < r < a$ and the other for $a < r < R$. On the other hand as the problem is axisymmetric, we consider a differential circular disc of size $2\pi rdr$.

Integrating within $0 < r < a$:

$$\int_0^a (\, p(r) - p_o) 2\pi rdr = \int_0^a \left(\frac{1}{2}\rho V^2 - \rho\frac{V^2}{8h^2}r^2\right) 2\pi rdr = 2\pi\left(\frac{1}{4}\rho V^2 a^2 - \frac{\rho V^2 a^4}{32h^2}\right)$$

Integrating within $a < r < R$:

$$\int_a^R (\, p(r) - p_o) 2\pi rdr = \int_a^R \left(\frac{1}{2}\rho V^2 - \rho\frac{a^4V^2}{8h^2r^2}\right) 2\pi rdr$$
$$= 2\pi\left(\frac{1}{4}\rho V^2(R^2 - a^2) - \frac{\rho a^4 V^2}{8h^2}\ln\left(\frac{R}{a}\right)\right)$$

Then adding both expressions and rearranging terms we get the expression:

$$\int_0^a (\, p(r) - p_o) 2\pi rdr + \int_a^R (\, p(r) - p_o) 2\pi rdr = F_p$$
$$= \boxed{\frac{1}{2}\pi V^2\left[R^2 - \frac{a^4}{8h^2}\right] + \pi\rho V^2\left(\frac{a^4}{4h^2}\right) ln\left(\frac{a}{R}\right)}$$

To explain why the pressure can be negative we look at indications of the exercise, these are that *R >> a and h < a*. If these statement are accomplished we come to the conclusion that the negative term within the expression gains weight ($\pi\rho V^2\left(\frac{a^4}{4h^2}\right) ln\left(\frac{a}{R}\right)$) being $ln\left(\frac{a}{R}\right)$ a negative number of considerable order, and $\left(\frac{a^4}{4h^2}\right)$ a positive number of considerable order but not as much as the other multiplying term.

By substituting the mentioned values in the expression (3) we get:

$$0.0981 = \frac{1}{2}\pi V^2\left[0.05^2 - \frac{0.01^4}{8 \cdot 0.001^2}\right] + \pi 1.2 V^2\left(\frac{0.01^4}{4 \cdot 0.001^2}\right) ln\left(\frac{0.01}{0.05}\right)$$

$$\boxed{V = 1.14\ \frac{m}{s}}$$

d) Once the force balance described above is achieved, the disk 'lifts off' from the lab bench that it is resting on and is attracted to the top flange plate, such that the gap decreases and $h = h(t)$. Use an appropriate form of conservation of mass to find new expressions (for $r \leq a$ and for $r \geq a$) for the unsteady velocity field $v(r, t)$ in the gap. Check that your expressions make sense physically and mathematically.

For deriving the expressions we will build a control volume for each case and impose that input flow must be equal to output flow plus the amount of volume that increases/decreases due to the moving disk.

For $r \leq a$:

$$\pi r^2 V - 2\pi r h V_r + \frac{dh}{dt}\pi r^2 = 0$$

$$\boxed{V_r = \frac{\pi r^2 V + \frac{dh}{dt}\pi r^2}{2\pi r h} = \frac{r\left(V + \frac{dh}{dt}\right)}{2h}}$$

For $r \geq a$:

$$\pi a^2 V - 2\pi r h V_r + \frac{dh}{dt}\pi r^2 = 0$$

$$\boxed{V_r = \frac{\pi a^2 V + \frac{dh}{dt}\pi r^2}{2\pi r h} = \frac{a^2 V + \frac{dh}{dt} r^2}{2rh}}$$

For both expressions we can check that for $dh/dt = 0$ we get the same solutions as the steady state case. We can also check that expressions are dimensionally homogeneous.

Exercise 2· Consider a two-dimensional uniform flow ($v = Uex$) past a cylinder of radium R located at the origin of coordinates- Assumes that the fluid is incompressible and inviscid, so that velocity can be defined using a stream function and Bernoulli's equation applies.

a) Which are the appropriate boundary conditions? How can they be expressed in terms of the stream function?

The fluid on the infinity is not being affected by the presence of the cylinder so $\boldsymbol{v} = U\boldsymbol{e}_x$.

$$\boxed{Vr(\infty, \theta) = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = U\cos\theta}$$

We also know that the cylinder is a solid surface, and then the flow cannot cross it.

$$\boxed{Vr(R, \theta) = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = 0}$$

b) Seek for a solution of the form

$$\psi(r,\theta) = f(r)\sin\theta \;\text{ with } f(r) = r^{\alpha}$$

For $\psi(r,\theta)$ being a valid stream function it is required that its Laplacian equals 0.

$$\nabla^2\psi(r,\theta) = 0$$

Applying the definition of Laplacian in polar coordinates and developing:

$$\frac{\partial^2\psi(r,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial\psi(r,\theta)}{\partial r} + + \frac{1}{r^2}\frac{\partial^2\psi(r,\theta)}{\partial\theta^2} = 0$$

$$(\alpha-1)\alpha r^{\alpha-2}\sin\theta + \frac{1}{r}\alpha r^{\alpha-1}\sin\theta + \frac{1}{r^2}(-r^{\alpha}\sin\theta) = 0$$

Taking common factor $r^{\alpha-2}\sin\theta$

$$r^{\alpha-2}\sin\theta\left((\alpha-1)\alpha + \alpha - 1\right) = 0$$

$$(\alpha-1)\alpha + \alpha - 1 = 0$$

Solving: $\alpha = \pm 1$, meaning that the flow is the combination of two flows with stream functions

$$\psi_1(r,\theta) = r^1\sin\theta$$

$$\psi_2(r,\theta) = r^{-1}\sin\theta$$

We can see that de form of the two stream functions are the same as the ones for a uniform flow and a line doublet, although it is necessary to scale those functions in order to meet the boundary conditions:

$$\psi_1(r,\theta) = C_1 r^1\sin\theta \;;\; \psi_2(r,\theta) = C_2 r^{-1}\sin\theta$$

Applying the boundary conditions:

$$at\; r = \infty \rightarrow \frac{1}{r}\frac{\partial\psi}{\partial\theta} = U\cos\theta \rightarrow \frac{1}{r}(C_1 r^1\cos\theta + C_2 r^{-1}\cos\theta) = U\cos\theta$$

$$(C_1\cos\theta + C_2\infty^{-2}\cos\theta) = U\cos\theta$$

$$C_1 = U$$

$$at\; r = R \rightarrow \frac{1}{r}\frac{\partial\psi}{\partial\theta} = 0 \rightarrow \frac{1}{r}(C_1 r^1\cos\theta + C_2 r^{-1}\cos\theta) = 0$$

$$(C_1\cos\theta + C_2 R^{-2}\cos\theta) = 0$$

$$U + C_2 R^{-2} = 0$$

$$C_2 = -UR^2$$

The stream function for the superposition of the two flows is:

$$\psi(r,\theta) = Ur^1\sin\theta - UR^2 r^{-1}\sin\theta$$

Rearranging

$$\boxed{\psi(r,\theta) = \mathrm{U}\sin\theta\left(\mathrm{r} - \frac{\mathrm{R}^2}{\mathrm{r}}\right)}$$

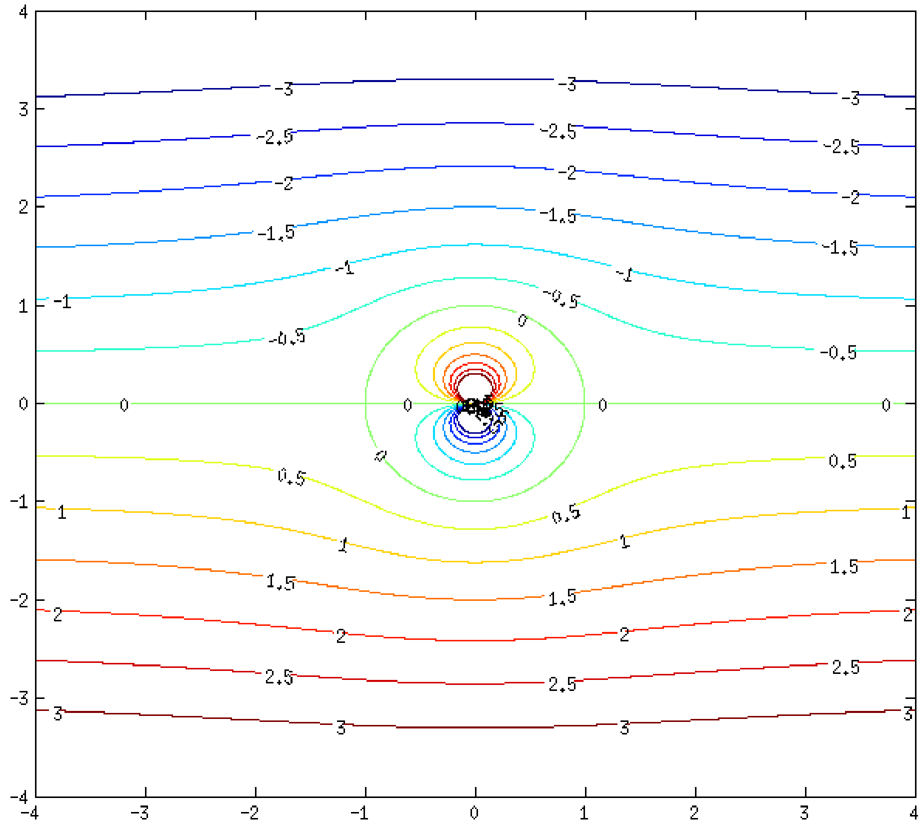


Figure 2. Plot of the stream function from ψ =-3 to ψ =3 in jumps of 0.5 $(with\ U = 1; r = 1)$

c) Compute the velocity field

In polar coordinates:

$$\mathrm{V_r} = \frac{1}{\mathrm{r}}\frac{\partial\psi}{\partial\theta} = \mathrm{U}\cos\theta\left(1 - \frac{\mathrm{R}^2}{\mathrm{r}^2}\right)$$

$$\mathrm{v_\theta} = -\frac{\partial\psi}{\partial\mathrm{r}} = -\mathrm{U}\sin\theta\left(1 + \frac{\mathrm{R}^2}{\mathrm{r}^2}\right)$$

In Cartesian coordinates:

$$u = \frac{\partial\psi}{\partial\mathrm{y}} = \frac{\partial}{\partial\mathrm{y}}\left(\mathrm{U\,y}\,(1 - \frac{\mathrm{R}^2}{\mathrm{x}^2+\mathrm{y}^2})\right) = \frac{U(R^2(y^2-x^2)+(x^2+y^2)^2)}{(x^2+y^2)^2}$$

$$v = -\frac{\partial\psi}{\partial\mathrm{x}} = -\frac{\partial}{\partial\mathrm{x}}\left(\mathrm{U\,y}\,(1 - \frac{\mathrm{R}^2}{\mathrm{x}^2+\mathrm{y}^2})\right) = -\frac{2R^2Uxy}{(x^2+y^2)^2}$$

d) Using Bernoulli's equation, compute the pressure field around the cylinder

Applying Bernoulli between a point on the cylinder's surface and a point far from it:

$$\frac{1}{2}V_\infty^2 + \frac{p_\infty}{\rho} + \cancel{gh_\infty} = \frac{1}{2}V_c^2 + \frac{p_c}{\rho} + \cancel{gh_c}$$

$$\frac{p_c - p_\infty}{\rho} = \frac{1}{2}(V_\infty^2 - V_c^2)$$

$$p_c - p_\infty = \frac{\rho}{2}\left(U^2 - \left(-\mathrm{U}\sin\theta\left(1 + \frac{\mathrm{R}^2}{\mathrm{R}^2}\right)\right)^2\right)$$

$$p_c - p_\infty = \frac{\rho}{2}(U^2 - (-2Usin\theta)^2)$$

$$p_c - p_\infty = \frac{\rho U^2}{2}(1 - 4sin^2\theta)$$

Taking into account the trigonometric identity $\sin^2 x = (1 - \cos{(2x)})/2$ we obtain:

$$\boxed{p_c - p_\infty = \frac{\rho U^2}{2}(\cos(2\theta) - 1)}$$

e) Compute the net force acting on the cylinder. Does this result make sense? Why?

$$\boldsymbol{F} = \int_0^{2\pi} p_c - p_\infty \, \boldsymbol{n} \, dA$$

Being the normal vector $\boldsymbol{n} = [cos\theta,\ sin\theta]$, and $dA = 1r\ d\theta$.

$$\boldsymbol{F} = \int_0^{2\pi} \frac{\rho U^2}{2}(\cos(2\theta) - 1)\ [cos\theta\,, sin\theta] r\ d\theta$$

Evaluating the integral:

$$\boxed{\boldsymbol{F} = [0, 0]}$$

As it was made the assumption of inviscid flow, the results obtained for the force could be expected. This is because the effects of viscosity are neglected causing the non-existance of a drag force and also the fact that the boundary layer never separates from the surface (no boundary layer).