Coupled Problems

Homeworks

1 Transmission conditions

1. The deflection $v(x)$ of an Euler-Bernouilli beam is governed by the differential equation

$$EI\frac{\mathrm{d}^4 v}{\mathrm{d}x^4} = f$$

where EI is a mechanical property of the beam section and the beam material and f is the distributed load. Assuming for example that the beam is clamped at $x = 0$ and $x = L$, the Principle of Virtual Work (PTV) states that the solution $v(x)$ satisfies

$$EI\int_0^L \frac{\mathrm{d}^2 \delta v}{\mathrm{d}x^2}\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = \int_0^L \delta v f$$

for all δv such that $\delta v(0) = \delta v(L) = 0$, $\frac{\mathrm{d}\delta v}{\mathrm{d}x}(0) = \frac{\mathrm{d}\delta v}{\mathrm{d}x}(L) = 0$.

   (a) Postulate the space of functions where both v and δv must belong. Justify the answer.

   (b) If $[0, L] = [0, P] \cup (P, L]$, obtain the transmission conditions at P implied by regularity requirements.

   (c) Obtain the transmission conditions at P that follow by imposing in the PTV that the integral is additive.

2. The Maxwell problem consists in finding a vector field $\boldsymbol{u} : \Omega \longrightarrow \mathbb{R}^3$ such that

$$\begin{aligned} \nu\nabla \times \nabla \times \boldsymbol{u} &= \boldsymbol{f} \quad \text{in } \Omega \\ \nabla \cdot \boldsymbol{u} &= 0 \quad \text{in } \Omega \\ \boldsymbol{n} \times \boldsymbol{u} &= \boldsymbol{0} \quad \text{on } \partial\Omega \end{aligned}$$

where $\nu > 0$, $\boldsymbol{f}$ is a divergence free force field and $\boldsymbol{n}$ the unit external normal. Equation $\nabla\cdot\boldsymbol{u} = 0$ is in fact redundant.

   (a) Write a variational statement of the problem. Postulate the space of functions where $\boldsymbol{u}$ must belong. Justify the answer.

   (b) If Γ is a surface that intersects Ω, obtain the transmission conditions across Γ implied by regularity requirements.

   (c) Obtain the transmission conditions across Γ that follow by imposing in the variational form of the problem that the integral is additive.

3. The Navier equations for an elastic material can be written in three different ways:

$$
\begin{aligned}
-2\mu\nabla\cdot(\boldsymbol{\varepsilon}(\boldsymbol{u}))-\lambda\nabla(\nabla\cdot\boldsymbol{u}) &= \rho\boldsymbol{b}\\
-\mu\Delta\boldsymbol{u}-(\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{u}) &= \rho\boldsymbol{b}\\
\mu\nabla\times(\nabla\times\boldsymbol{u})-(\lambda+2\mu)\nabla(\nabla\cdot\boldsymbol{u}) &= \rho\boldsymbol{b}
\end{aligned}
$$

where $\boldsymbol{u}$ is the displacement field, $\boldsymbol{\varepsilon}(\boldsymbol{u})$ the symmetric part of $\nabla\boldsymbol{u}$, λ and μ the Lamé coefficients, ρ the density of the material and $\boldsymbol{b}$ the body forces. Let us assume that $\boldsymbol{u}=\boldsymbol{0}$ on $\partial\Omega$.

   (a) Write down the variational form of the previous equations in the appropriate functional spaces.

   (b) If Γ is a surface that intersects Ω, obtain the transmission conditions across Γ that follow by imposing in the variational form of the problem that the integral is additive.

2 Domain decomposition methods

1. Consider Problem 1 of Section 1. Let $[0,L]=[0,L_1]\cup[L_2,L]$, with $L_2<L_1$.

   (a) Write down an iteration-by-subdomain scheme based on a Schwarz additive domain decomposition method.

   (b) Obtain the matrix version of the previous scheme once space has been discretized using finite elements.

2. Consider Problem 2 of Section 1. Let Γ be a surface that intersects Ω.

   (a) Write down an iteration-by-subdomain scheme based on the Dirichlet-Neumann coupling.

   (b) Obtain the expression of the Steklov-Poincaré operator of the problem.

   (c) Obtain the matrix version of the previous scheme once space has been discretized using finite elements.

3. Consider the problem of finding $u:\Omega\longrightarrow\mathbb{R}$ such that

$$
\begin{aligned}
-k\Delta u &= f \quad \text{in } \Omega\\
u &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

where $k>0$. Let Γ be a surface crossing Ω.

   (a) Write down an iteration-by-subdomain scheme based on the Dirichlet-Robin coupling.

   (b) Obtain the matrix version of the previous scheme once space has been discretized using finite elements.

   (c) Obtain the Schur complement as discrete version of the Steklov-Poincaré operator.

   (d) Identify the preconditioner for the Schur complement equation arising from the iterative scheme of section (a).

3 Coupling of heterogeneous problems

1. Consider the beam described in Problem 1 of Section 1. Apart from being clamped at $x = 0$ and $x = L$, the beam is supported on an elastic wall that occupies the square $[0, L] \times [-L, 0]$, where $y = 0$ corresponds to the beam axis. The wall is clamped everywhere except on the upper wall, where the beam is. The wall displacements in the x- and y-directions are u and v, respectively, and the elastic properties E (Young modulus) and ν (Poisson's coefficient). No loads are applied on the wall, except for those coming from the beam.

   (a) Write down the equations in the wall assuming a plane stress behavior.

   (b) Write down the equations for the beam modified because of the presence of the wall.

   (c) Obtain the adequate transmission conditions for v and the normal component of the traction on the wall at $y = 0$.

   (d) *Suggest* transmission conditions for u and the tangent component of the traction on the wall at $y = 0$. Discuss the implications if this component is not assumed to be zero.

2. Let $\mathcal{S}_D$ and $\mathcal{S}_S$ be the Dirichlet-to-Neumann operators for the Darcy and the Stokes problems, respectively (see the class notes, chapter 3). The Steklov-Poincaré equation can be written as

$$\mathcal{S}_S(\lambda) = \mathcal{S}_D(\lambda),$$

   where λ is the normal velocity on Γ, the interface between the Darcy and the Stokes regions.

   (a) Obtain the discrete version of the previous equation when space is discretized using finite elements. Relate the resulting matrices to those arising from the discretization of the Darcy and the Stokes problems separately.

   (b) Write down the matrix form of a Dirichlet-Neumann iteration-by-subdomain using the matrices of the Darcy and the Stokes problems.

   (c) Identify the Richardson iteration for the algebraic problem in (a) resulting from (b).

4 Monolithic and partitioned schemes in time

Consider the one-dimensional, transient, heat transfer equation:

$$\begin{aligned}
\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} &= f \quad \text{in [0,1]} \\
u(x = 0, t) &= 0 \\
u(x = 1, t) &= 0 \\
u(x, t = 0) &= 0
\end{aligned}$$

1. Discretize it using the finite element method (linear elements, element size h) for the discretization in space, and a BDF1 scheme for the discretization in time. Write down the weak form of the problem and the resulting matrix form of the problem, including the corresponding boundary integrals if necessary. Consider $\kappa = 1$, $f = 1$, $\delta t = 1$.

2. Consider a domain decomposition approach for the previous problem. The left subdomain is composed of 2 elements (h = 0.2), while the right subdomain is composed of 3 elements (h = 0.2). Show that, if a monolithic approach is adopted, no boundary integrals are required at the interface. From now on, we denote the values at the nodes of the mesh as u_0, u_1, u_2, u_3, u_4, u_5. The interface is at u_2.

3. Obtain the algebraic form of the Dirichlet-to-Neumann operator (Steklov-Poincaré's operator) for the left subdomain, departing from given values of u_i^n at time step n, and an interface value u_2^{n+1}.

4. Obtain the algebraic form of the Neumann-to-Dirichlet operator for the right subdomain, departing from given values of u_i^n and an interface value for the fluxes $\phi^{n+1} = \kappa \partial_x u^{n+1}$ at the coordinate of node 2.

5. Write down the iterative algorithm for a staggered approach applying Dirichlet boundary conditions at the interface to the left subdomain and Neumann boundary conditions at the interface for the right subdomain.

6. Do the same for a substitution and an iteration by subdomains scheme.

7. Rewrite the algebraic system associated to the left subdomain (Dirichlet boundary conditions at the interface), using Nitsche's method for applying the boundary conditions. How does the condition number of the resulting system of equations vary with the penalty parameter α?

5 Operator splitting techniques

Consider the one dimensional, transient, convection-diffusion equation:

$$\begin{aligned} \frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} + a_x \frac{\partial u}{\partial x} &= f \quad \text{in [0,1]} \\ u(x=0,t) &= 0 \\ u(x=1,t) &= 0 \\ u(x,t=0) &= 0 \end{aligned}$$

with $\kappa = 1, a_x = 1, f = 1$.

1. Discretize it in space using finite elements (2 elements) and in time (finite differences, BDF1). Solve the first step of the problem, writing the solution as a function of the time step size δt.

2. Solve the same time step by using a first order operator splitting technique.

3. Evaluate the error of the splitting approach with respect to the monolithic approach. Plot the splitting error vs. the time step size for $\delta t = 1$, $\delta t = 0.5$, $\delta t = 0.25$. Comment on the results.

6 Fractional step methods

Consider the fractional step approach for the incompressible Navier-Stokes equations (Yosida scheme):

$$\begin{aligned} & M\frac{1}{\delta t}\left(\hat{U}^{n+1} - U^n\right) + K\hat{U}^{n+1} = f - G\tilde{P}^{n+1} \\ & DM^{-1}GP^{n+1} = \frac{1}{\delta t}D\hat{U}^{n+1} - DM^{-1}G\tilde{P}^{n+1} \\ & M\frac{1}{\delta t}\left(U^{n+1} - \hat{U}^{n+1}\right) + \alpha K\left(U^{n+1} - \hat{U}^{n+1}\right) + G\left(P^{n+1} - \tilde{P}^{n+1}\right) = 0 \end{aligned}$$

1. Which is the optimal value for the α parameter?

2. What is the source of error of the scheme?

7 ALE formulations

1. Given the spatial description of a property

$$\gamma(x, y, z, t) = \left[2x, ye^t, z\right]$$

the equations of movement:

$$\begin{aligned} x &= Xe^t \\ y &= Y + e^t - 1 \\ z &= Z \end{aligned}$$

and the equations of the movement of the mesh:

$$\begin{aligned} x_m &= \mathcal{X} + \alpha t \\ y_m &= \mathcal{Y} - \beta t \\ z_m &= \mathcal{Z} \end{aligned}$$

   (a) Obtain the description of the property in terms of the ALE coordinates $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$.
   (b) Compute the velocity of the particles and the mesh velocity.
   (c) Compute the ALE description of the material temporal derivative of γ.

2. Write down the ALE form of the incompressible Navier-Stokes equations. Where (in time and space) is each of the terms of the equation evaluated? How are temporal derivatives computed?

3. Do a bibliographical research on existing methods for the definition of the mesh movement in ALE formulations (Poisson problem, Elasticity problem, etc.). Describe the main advantages of each of these methods.

8 Fluid-Structure Interaction

1. Describe the added mass effect problem for fluid structure interaction problems. When does it appear, what kind of problems suffer from it? What are the main methods for dealing with it?

2. Consider the iteration by subdomain scheme for the heat transfer problem described in problem 1. Apply 2 iterations of the AITKEN relaxation scheme to it.

3. Consider the monolithic (1 domain), transient (BDF1), finite element (linear elements, $h = 1/4$) approximation of the heat transfer equation in problem 1. Enforce the Dirichlet boundary conditions in $x = 0$ and $x = 1$ by using Lagrange multipliers. What is the form of the discrete system? What is the condition number of the resulting matrix?

4. Consider the monolithic (1 domain), transient (BDF1), finite element (linear elements, $h = 1/4$) approximation of the heat transfer equation in problem 1. Suppose that a level set function ($\psi = 0$ at $x = 0.4$) divides the domain into a high thermal conductivity ($\kappa = 100$) subdomain ($x \in [0, 0.4]$) and a low thermal conductivity ($\kappa = 1$) subdomain ($x \in (0.4, 1]$). Build the system matrix for this problem. Take into account the need for subintegrating the element cut by the level set function.