FLUID-STRUCTURE INTERACTION

INTRODUCTION

Fluid-Structure Interaction is one of most well-known type of coupled problems.
It is present in a number of engineering problems of interest:

- Aircraft engineering

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- Civil Engineering Bridge design

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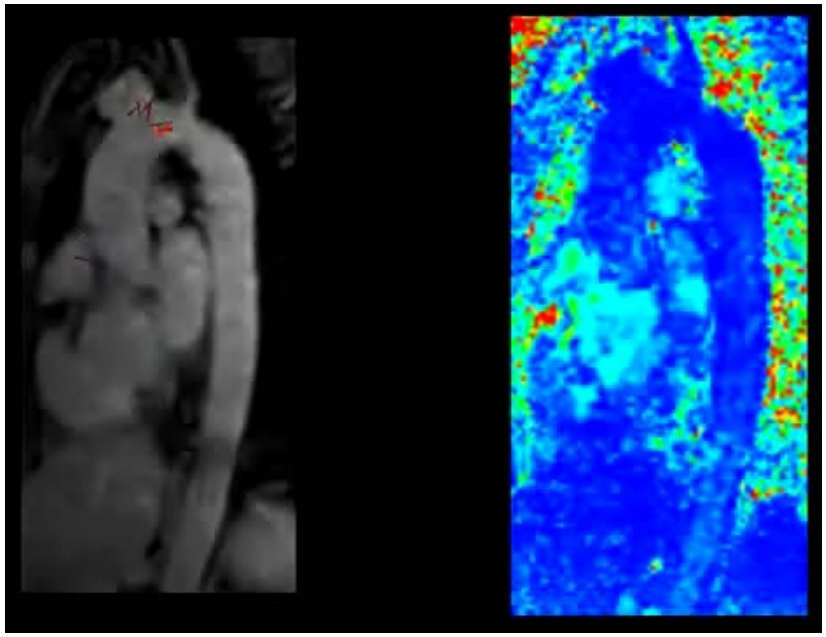
- Biomedical engineering



Flow in the aorta, magnetic resonance imaging

The added mass effect is particularly challenging in this kind of problems

INTRODUCTION

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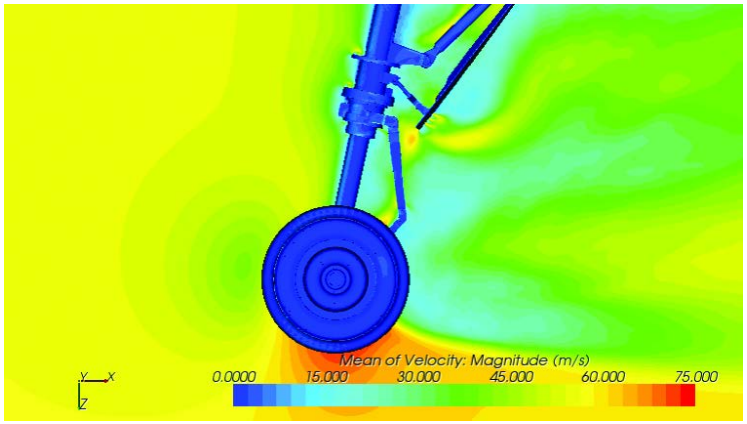
It is present in a number of engineering problems of interest:

- Aeroacoustics Fluid-Structure Interaction



INTRODUCTION

We can classify FSI problems into:

- One way FSI problems:

  The solid body has a prescribed movement, which induces a movement in the fluid.

  These problems are easy to solve, this is almost like two decoupled problems.

- Two way FSI problems:

  The flow around a solid body induces a movement in in the solid body through the application of forces.

  The movement of the solid body changes the boundary conditions for the fluid problem, which in turn affects the characteristics of the flow around the object.

PROBLEM STATEMENT

- (Incompressible) Navier-Stokes equations

$$\partial_t \boldsymbol{u}_f + \boldsymbol{u}_f \cdot \nabla \boldsymbol{u}_f - \nu \Delta \boldsymbol{u}_f + \nabla p = \boldsymbol{f} \qquad \text{on } \Omega_f$$
$$\nabla \cdot \boldsymbol{u}_f = 0 \qquad \text{on } \Omega_f$$

with boundary and initial conditions:

$$\boldsymbol{u}_f = \bar{\boldsymbol{u}} \text{ on } \Gamma_{Df} \qquad \boldsymbol{n} \cdot \boldsymbol{\sigma}_f = \boldsymbol{t} \text{ on } \Gamma_{Nf} \qquad \boldsymbol{u}_f(\boldsymbol{x}, 0) = \boldsymbol{u}_{0f}$$

- Solid mechanics dynamic equations

$$\rho \partial_t \boldsymbol{u}_s = \nabla \cdot \boldsymbol{\sigma}_s + \rho \boldsymbol{b} \qquad \text{on } \Omega_s$$
$$\rho_s = \rho_0 J \qquad \text{on } \Omega_s$$
$$J = \det|\boldsymbol{F}| \qquad \boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}$$

with boundary and initial conditions:

$$\boldsymbol{u}_s = \bar{\boldsymbol{u}}_s \text{ on } \Gamma_{Ds} \qquad \boldsymbol{n} \cdot \boldsymbol{\sigma}_s = \boldsymbol{t}_s \text{ on } \Gamma_{Ns} \qquad \boldsymbol{u}_s(\boldsymbol{x}, 0) = \boldsymbol{u}_{0s}$$

- Transmission conditions

$$\boldsymbol{u}_s = \boldsymbol{u}_f \qquad \text{on } \Gamma_{\text{interface}}$$
$$\boldsymbol{n} \cdot \boldsymbol{\sigma}_s = \boldsymbol{n} \cdot \boldsymbol{\sigma}_f \qquad \text{on } \Gamma_{\text{interface}}$$

GENERAL APPROACH FOR THE SOLUTION OF THE COUPLED PROBLEM

The most usual approach for dealing with FSI problems is:

- Solid mechanics problem is treated in a purely Lagrangian way.
- Fluid mechanics problem: Arbitrary Lagrangian Eulerian formulation.

$$\partial_t \boldsymbol{u}_f + (\boldsymbol{u}_f - \boldsymbol{u}_{MESH}) \cdot \nabla \boldsymbol{u}_f - \nu \Delta \boldsymbol{u}_f + \nabla p = \boldsymbol{f} \qquad \text{on } \Omega_f$$

- Non-overlapping domain decomposition methods.

- Transmission conditions
  - Dirichlet/Neumann
  - Robin-Robin

- Coupling in time:
  - Monolithic
  - Partitioned

    Added mass effect for incompressible flows

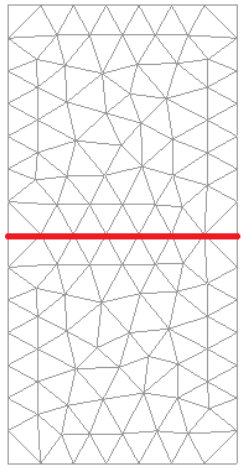
MONOLITHIC APPROACH

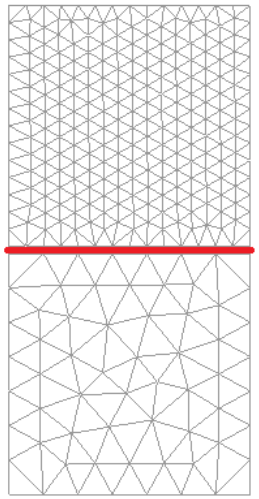
Transmission conditions in the Monolithic approach:

- Dirichlet (Fluid): The fluid velocity is set to be that of the solid
- Neumann (Solid): we transfer the stress in the solid body surface and we apply it to the fluid.

Robin – Robin conditions are also possible: Any kind of transmission conditions will result in an equivalent problem.

Grids can either be matching or non-matching:





MONOLITHIC APPROACH

Let us consider a Neumann (Solid) – Neumann (Fluid) approach.

- The equations for the solid are (general, before coupling):

$$(\boldsymbol{v}_h^s, \partial_t \boldsymbol{u}_h^s)_{\Omega^s} + (\boldsymbol{v}_h^s, \sigma(u_h{}^s)^s)_{\Omega^s} - < \boldsymbol{v}_h^s, \boldsymbol{n} \cdot \boldsymbol{\sigma}(\boldsymbol{u}_h{}^s)^s >_{\Gamma_{\text{interface}}} = < \boldsymbol{v}_h^s, \rho \boldsymbol{b} >_{\Omega^s}$$

- On the other hand, we have the fluid equation with: $\boldsymbol{n} \cdot \boldsymbol{\sigma}\left(\boldsymbol{u}_h^f\right)^f = \boldsymbol{n} \cdot \boldsymbol{\sigma}(\boldsymbol{u}_h^s)^s$

$$(\boldsymbol{v}_h^f, \partial_t \boldsymbol{u}_h^f)_{\Omega^f} + \left(\nabla \boldsymbol{v}_h^f, \sigma\left(u_h{}^f\right)^f\right)_{\Omega^f} - < \boldsymbol{v}_h^f, \boldsymbol{n}^f \cdot \boldsymbol{\sigma}(\boldsymbol{u}_h)^s >_{\Gamma_{\text{interface}}} = < \boldsymbol{v}_h^f, \rho \boldsymbol{b} >_{\Omega^f}$$

If the meshes match at the interface and we are using the same interpolation space V_h for $\boldsymbol{u}_h^s$ and $\boldsymbol{u}_h^f$:

$$- < \boldsymbol{v}_h^f, \boldsymbol{n}^f \cdot \boldsymbol{\sigma}(\boldsymbol{u}_h^s)^s >_{\Gamma_{\text{interface}}} = < \boldsymbol{v}_h^s, \boldsymbol{n}^s \cdot \boldsymbol{\sigma}(\boldsymbol{u}_h{}^s)^s >_{\Gamma_{\text{interface}}} =$$
$$(\boldsymbol{v}_h^s, \partial_t \boldsymbol{u}_h^s)_{\Omega^s} + (\boldsymbol{v}_h^s, \sigma(u_h{}^s)^s)_{\Omega^s} - < \boldsymbol{v}_h^s, \rho \boldsymbol{b} >_{\Omega^s}$$

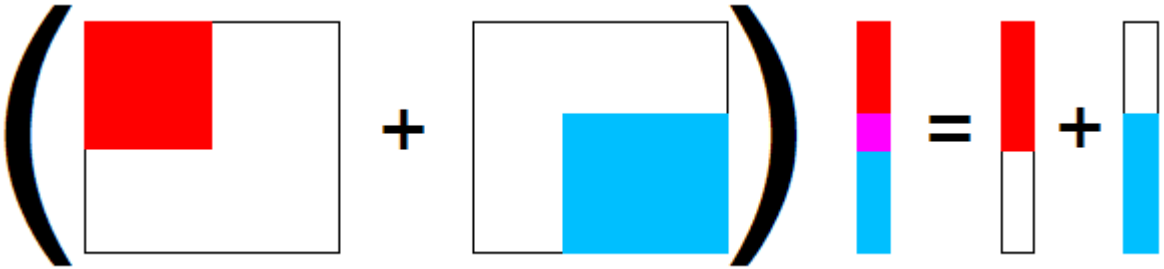
MONOLITHIC APPROACH

So we can write the problem for the fluid domain as:

$$\left(\boldsymbol{v}_h^f, \partial_t \boldsymbol{u}_h^f\right)_{\Omega^f} + \left(\nabla \boldsymbol{v}_h^f, \sigma\left(u_h{}^f\right)^f\right)_{\Omega^f} + \left(\boldsymbol{v}_h^s, \partial_t \boldsymbol{u}_h^s\right)_{\Omega^s} + \left(\boldsymbol{v}_h^s, \sigma(u_h{}^s)^s\right)_{\Omega^s} =$$
$$< \boldsymbol{v}_h^f, \rho \boldsymbol{b} >_{\Omega^f} + < \boldsymbol{v}_h^s, \rho \boldsymbol{b} >_{\Omega^s}$$

The equations for the solid domain are exactly the same.

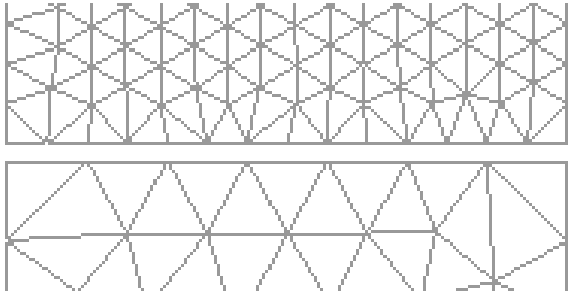
- There is no need to integrate boundary terms.
- We can simply assemble the global system and use the unknowns at the interface to account for both fluid and solid velocity values.



MONOLITHIC APPROACH

If the grids are non-matching this is not possible. We need to have unknowns for both the fluid and the solid.

An interpolation of the $\sigma(u_h{}^s)^s$, $\sigma\left(u_h{}^f\right)^f$, $\boldsymbol{u}_h^s$, $\boldsymbol{u}_h^f$ values is required at the Gauss points.



The final system for non-matching grids is (supposing a Neumann-Neumann scheme):

$$(\boldsymbol{v}_h^s, \partial_t \boldsymbol{u}_h^s)_{\Omega^s} + (\boldsymbol{v}_h^s, \sigma(u_h{}^s)^s)_{\Omega^s} - < \boldsymbol{v}_h^s, \boldsymbol{n^s} \cdot \boldsymbol{\sigma}\left(\boldsymbol{u}_h{}^f\right)^f >_{\Gamma_{\text{interface}}} = < \boldsymbol{v}_h^s, \rho \boldsymbol{b} >_{\Omega^s}$$

$$\left(\boldsymbol{v}_h^f, \partial_t \boldsymbol{u}_h^f\right)_{\Omega^{\text{f}}} + \left(\nabla \boldsymbol{v}_h^f, \sigma\left(u_h{}^f\right)^f\right)_{\Omega^{\text{f}}} - < \boldsymbol{v}_h^f, \boldsymbol{n^f} \cdot \boldsymbol{\sigma}(\boldsymbol{u}_h)^s >_{\Gamma_{\text{interface}}} = < \boldsymbol{v}_h^f, \rho \boldsymbol{b} >_{\Omega^{\text{f}}}$$

PARTITIONED APPROACHES

Partitioned approaches are possible as for any other coupled problem.

If the grids are matching, the Neumann boundary terms can be computed as in the monolithic case.

However, this approach is not usually pursued because:

- We want to take advantage of the possibility of using different mesh refinements in the fluid and the solid.
- Many times the fluid and the solid are solved by using different computational software: we do not know the interpolation space nor the forcing term in the other subdomain (piece of software).

PARTITIONED APPROACHES

Many times explicit partitioning schemes a la Gauss-Seidel (staggered appoach) are used:

1. Solve for $\boldsymbol{u}_h^{s,n+1}$:

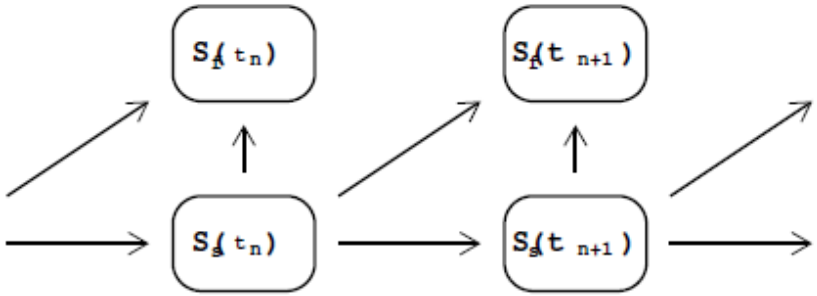
$$\left(\boldsymbol{v}_h^s, \partial_t \boldsymbol{u}_h^{s,n+1}\right)_{\Omega^s} + \left(\boldsymbol{v}_h^s, \sigma\left(\boldsymbol{u}_h^{s,n+1}\right)^s\right)_{\Omega^s} = < \boldsymbol{v}_h^s, \rho \boldsymbol{b} >_{\Omega^s}$$

$$\boldsymbol{u}_h^{s,n+1} = \boldsymbol{u}_h^{f,n} \quad \text{on } \Gamma_{interface}$$

2. Solve for $\boldsymbol{u}_h^{f,n+1}$:

$$\left(\boldsymbol{v}_h^f, \partial_t \boldsymbol{u}_h^{f,n+1}\right)_{\Omega^f} + \left(\nabla \boldsymbol{v}_h^f, \sigma\left(u_h{}^{f,n+1}\right)^f\right)_{\Omega^f} - < \boldsymbol{v}_h^f, \boldsymbol{n}^f \cdot \boldsymbol{\sigma}\left(\boldsymbol{u}_h^{s,n+1}\right)^s >_{\Gamma_{\text{interface}}} =$$

$$< \boldsymbol{v}_h^f, \rho \boldsymbol{b} >_{\Omega^f}$$



From Tomas Richter. Numerical Methods for FSI

PARTITIONED APPROACHES

Due to the enhanced parallel capabilities, sometimes a la Jacobi schemes are used (partitioned approach):

1. Solve for $\boldsymbol{u}_h^{s,n+1}$:

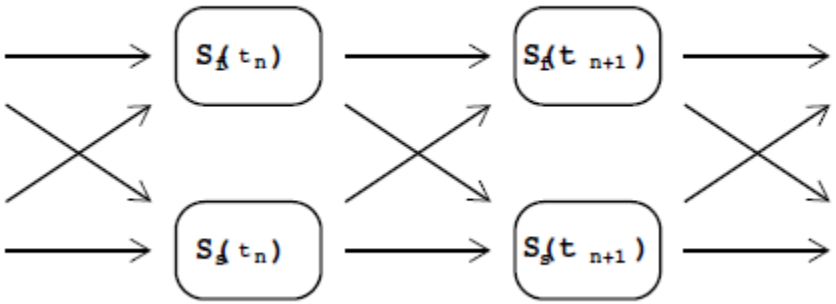
$$\left(\boldsymbol{v}_h^s, \partial_t \boldsymbol{u}_h^{s,n+1}\right)_{\Omega^s} + \left(\boldsymbol{v}_h^s, \sigma\left(\boldsymbol{u}_h^{s,n+1}\right)^s\right)_{\Omega^s} = < \boldsymbol{v}_h^s, \rho \boldsymbol{b} >_{\Omega^s}$$

$$\boldsymbol{u}_h^{s,n+1} = \boldsymbol{u}_h^{f,n} \quad \text{on } \Gamma_{interface}$$

2. Solve for $\boldsymbol{u}_h^{f,n+1}$:

$$\left(\boldsymbol{v}_h^f, \partial_t \boldsymbol{u}_h^{f,n+1}\right)_{\Omega^f} + \left(\nabla \boldsymbol{v}_h^f, \sigma\left(u_h{}^{f,n+1}\right)^f\right)_{\Omega^f} - < \boldsymbol{v}_h^f, \boldsymbol{n}^f \cdot \boldsymbol{\sigma}\left(\boldsymbol{u}_h^{s,n}\right)^s >_{\Gamma_{\text{interface}}} =$$

$$< \boldsymbol{v}_h^f, \rho \boldsymbol{b} >_{\Omega^f}$$



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PARTITIONED APPROACHES

Strongly coupled partitioned approaches are also possible. In these approaches we iterate withing each time step until we reach convergence:

1. Solve for $\boldsymbol{u}_h^{s,n+1,i+1}$:

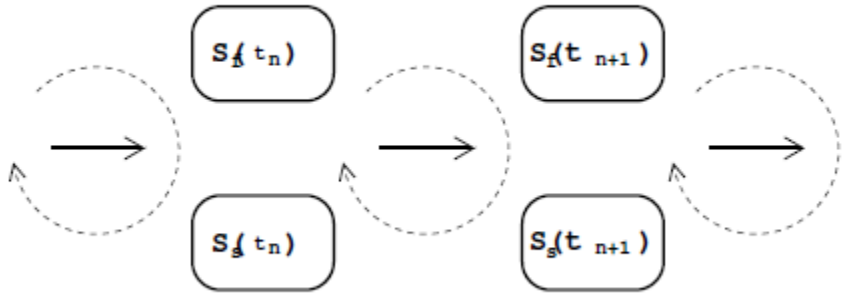
$$\left(\boldsymbol{v}_h^s, \partial_t \boldsymbol{u}_h^{s,n+1.i+1}\right)_{\Omega^s} + \left(\boldsymbol{v}_h^s, \sigma\left(\boldsymbol{u}_h^{s,n+1,i+1}\right)^s\right)_{\Omega^s} = < \boldsymbol{v}_h^s, \rho \boldsymbol{b} >_{\Omega^s}$$

$$\boldsymbol{u}_h^{s,n+1,i+1} = \boldsymbol{u}_h^{f,n+1,i} \quad \text{on } \Gamma_{interface}$$

2. Solve for $\boldsymbol{u}_h^{f,n+1}$:

$$\left(\boldsymbol{v}_h^f, \partial_t \boldsymbol{u}_h^{f,n+1,i+1}\right)_{\Omega^f} + \left(\nabla \boldsymbol{v}_h^f, \sigma\left(u_h{}^{f,n+1,i+1}\right)^f\right)_{\Omega^f} -$$

$$< \boldsymbol{v}_h^f, \boldsymbol{n}^f \cdot \boldsymbol{\sigma}\left(\boldsymbol{u}_h^{s,n+1,i}\right)^s >_{\Gamma_{\text{interface}}} = < \boldsymbol{v}_h^f, \rho \boldsymbol{b} >_{\Omega^f}$$



Depending on the number of iterations, this could be slower than using the monolithic approach!

PARTITIONED APPROACHES

However, when dealing with incompressible flow, partitioned approaches (explicit and iterating within a time step) do not always converge.

This is due to the so-called **added-mass effect.**

Let us study a simplified problem:

- Viscous effects are neglected
- The convective term in the Navier-Stokes equations is dropped
- ALE effects are not taken into account
- There are no external forces, nor Dirichlet boundary conditions

$$(\boldsymbol{v}_h^s, \partial_t \boldsymbol{u}_h^s)_{\Omega^s} + (\boldsymbol{v}_h^s, \sigma(u_h{}^s)^s)_{\Omega^s} - < \boldsymbol{v}_h^s, \boldsymbol{f}^s >_{\Gamma_{\text{interface}}} = \mathbf{0}$$
$$(\boldsymbol{v}_h^f, \partial_t \boldsymbol{u}_h^f)_{\Omega^f} - (\nabla \cdot \boldsymbol{u}_h, p)_{\Omega^f} = < \boldsymbol{v}_h, \boldsymbol{f}^f >_{\Gamma_{\text{interface}}}$$
$$\left(\boldsymbol{v}_h^f, \nabla \cdot \boldsymbol{u}_h\right)_{\Omega_f} = 0$$

When the iterative scheme has converged:

$$< \boldsymbol{v}_h, \boldsymbol{f}^f >_{\Gamma_{\text{interface}}} = < \boldsymbol{v}_h^s, \boldsymbol{f}^s >_{\Gamma_{\text{interface}}}$$

PARTITIONED APPROACHES

The discrete system is now:

$$\rho_f \boldsymbol{M}^f \boldsymbol{u_f}' + \boldsymbol{B}p = \boldsymbol{F}_\Gamma^f$$
$$\boldsymbol{B^T u_f} = 0$$

$$\boldsymbol{\rho_s M^s d_s''} + \mathbf{K}\boldsymbol{d_s} = \boldsymbol{F}_\Gamma^s$$

We note that if we neglect the ALE terms $\boldsymbol{B^T u_f} = 0$ is equivalent to:

$$\boldsymbol{B^T u_f}' = 0$$

And we have:

$$\rho_f \boldsymbol{M}^f \boldsymbol{u_f}' + \boldsymbol{B}p = \boldsymbol{F}_\Gamma^f$$
$$\boldsymbol{B^T u_f}' = 0$$

We split $\boldsymbol{u_f}$ into boundary and interior parts:

$$\begin{bmatrix} \rho_f \boldsymbol{M}^f_{\Omega\Omega} & \rho_f \boldsymbol{M}^f_{\Omega\Gamma} & \boldsymbol{B}_\Omega \\ \rho_f \boldsymbol{M}^f_{\Gamma\Omega} & \rho_f \boldsymbol{M}^f_{\Gamma\Gamma} & \boldsymbol{B}_\Gamma \\ \boldsymbol{B}^T_\Omega & \boldsymbol{B}^T_\Gamma & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u_{f\Omega}}' \\ \boldsymbol{u_{f\Gamma}}' \\ p \end{bmatrix} = \begin{bmatrix} 0 \\ \boldsymbol{F}^f_\Gamma \\ 0 \end{bmatrix}$$

PARTITIONED APPROACHES

We approximate the mass matrix by a lumped mass matrix:

$$\boldsymbol{M}^f = |K|\boldsymbol{I}^f$$

Off diagonal terms of the mass matrix disappear due to lumping:

$$\begin{bmatrix} \rho_f|K|\boldsymbol{I}^f_{\Omega\Omega} & 0 & \boldsymbol{B}_\Omega \\ 0 & \rho_f|K|\boldsymbol{I}^f_{\Gamma\Gamma} & \boldsymbol{B}_\Gamma \\ \boldsymbol{B}^T_\Omega & \boldsymbol{B}^T_\Gamma & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_{f\Omega}{}' \\ \boldsymbol{u}_{f\Gamma}{}' \\ p \end{bmatrix} = \begin{bmatrix} 0 \\ \boldsymbol{F}^f_\Gamma \\ 0 \end{bmatrix}$$

Assuming $\boldsymbol{u}'_{f\Gamma}$ is known, using the third equation and replacing in the first one, we can express the remaining fluid variables as:

$$p = \rho_f|K|\left(\boldsymbol{B}^T_\Omega\boldsymbol{B}_\Omega\right)^{-1}\boldsymbol{B}^T_\Gamma\boldsymbol{u}'_{f\Gamma}$$

$$\boldsymbol{u}'_{f\Omega} = -\boldsymbol{B}_\Omega\left(\boldsymbol{B}^T_\Omega\boldsymbol{B}_\Omega\right)^{-1}\boldsymbol{B}^T_\Gamma\boldsymbol{u}'_{f\Gamma}$$

Now we can express the resulting boundary force as:

$$\boldsymbol{F}^f_\Gamma = \rho_f|K|\boldsymbol{I}^f_{\Gamma\Gamma}\boldsymbol{u}'_{f\Gamma} + \boldsymbol{B}_\Gamma p = \rho_f|K|\left[\boldsymbol{I}^f_{\Gamma\Gamma} + \left(\boldsymbol{B}^T_\Omega\boldsymbol{B}_\Omega\right)^{-1}\boldsymbol{B}^T_\Gamma\right]\boldsymbol{u}'_{f\Gamma}$$

$\left[\boldsymbol{I}^f_{\Gamma\Gamma} + \left(\boldsymbol{B}^T_\Omega\boldsymbol{B}_\Omega\right)^{-1}\boldsymbol{B}^T_\Gamma\right]$ is called the added mass operator $\boldsymbol{M}_A$. It allows to express the new interface forces for the solid as:

$$\boldsymbol{F}^S_\Gamma = \rho_f|K|\boldsymbol{M}_A\boldsymbol{u}'_{f\Gamma}$$

PARTITIONED APPROACHES

In the solid system, the added mass operator acts as an additional mass in the interface nodes:

$$\rho_S \begin{bmatrix} \boldsymbol{M}^S_{\Omega\Omega} & \boldsymbol{M}^S_{\Omega\Gamma} \\ \boldsymbol{M}^S_{\Gamma\Omega} & \boldsymbol{M}^S_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \boldsymbol{d}''_{\boldsymbol{\Omega}} \\ \boldsymbol{d}''_{\boldsymbol{\Gamma}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{K}^S_{\Omega\Omega} & \boldsymbol{K}^S_{\Omega\Gamma} \\ \boldsymbol{K}^S_{\Gamma\Omega} & \boldsymbol{K}^S_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \boldsymbol{d}_{\boldsymbol{\Omega}} \\ \boldsymbol{d}_{\boldsymbol{\Gamma}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ -\rho_f |K| \boldsymbol{M}_A \boldsymbol{u}'_{\boldsymbol{f\Gamma}} \end{bmatrix}$$

Let us now introduce an explicit time discretization scheme and take into account that the accelerations of the fluid and the solid coincide at the interface:

$$\boldsymbol{u}'_{\boldsymbol{f\Gamma}}{}^{n+1} = \boldsymbol{d}''_{\Gamma}{}^{n+1} = \frac{1}{\Delta t^2}\left(\boldsymbol{d}_{\Gamma}{}^{n} - 2\boldsymbol{d}_{\Gamma}{}^{n-1} + \boldsymbol{d}_{\Gamma}{}^{n-2}\right)$$

The solid body force at iteration *i* is given by:

$$\boldsymbol{F}_{\Gamma}^{s,i,n+1} = \frac{1}{\Delta t^2}\rho_f |K| \boldsymbol{M}_A \left(\boldsymbol{d}_{\Gamma}{}^{n} - 2\boldsymbol{d}_{\Gamma}{}^{n-1} + \boldsymbol{d}_{\Gamma}{}^{n-2}\right)$$

We assume small time steps so that we can neglect the stiffness matix in the solid. Plus, we consider a lumped mass matrix:

$$\rho_s |S| I^s \boldsymbol{d}''_{\boldsymbol{\Gamma}} = -\rho_f |K| \boldsymbol{M}_A \boldsymbol{u}'_{\boldsymbol{f\Gamma}}$$

PARTITIONED APPROACHES

$$\rho_s|S|I^s \frac{1}{\Delta t^2}\left(\boldsymbol{d}_\Gamma{}^{n+1} - 2\boldsymbol{d}_\Gamma{}^{n} + \boldsymbol{d}_\Gamma{}^{n-1}\right)$$
$$= -\rho_f|K|\boldsymbol{M}_A \frac{1}{\Delta t^2}\left(\boldsymbol{d}_\Gamma{}^{n} - 2\boldsymbol{d}_\Gamma{}^{n-1} + \boldsymbol{d}_\Gamma{}^{n-2}\right)$$

We note that $\boldsymbol{M}_A$ is positive definite and symmetric. We can diagonalize it (with eigenvalues μ_i) an write the system for each component of the basis independently:

$$\boldsymbol{d}_\Gamma = \sum d_i \boldsymbol{D}_i$$

And then:

$$\left(d_i{}^{n+1} - 2d_i{}^{n} + d_i{}^{n-1}\right) + \frac{\rho_f|K|}{\rho_s|S|}\mu_i\left(d_i{}^{n} - 2d_i{}^{n-1} + d_i{}^{n-2}\right) = 0.$$

We write $d_i{}^{n+1} = \lambda_i d_i^n$ with λ_i the amplification factors and then:

$$\left(\lambda_i^3 - 2\lambda_i^2 + \lambda_i^3\right) + \frac{\rho_f|K|}{\rho_s|S|}\mu_i\left(\lambda_i^2 - 2\lambda_i + 1\right) = 0.$$

This results in $\lambda_1 = \lambda_2 = 1$ and

$$\lambda_3 = -\frac{\rho_f|K|}{\rho_s|S|}\mu_i$$

PARTITIONED APPROACHES

For the scheme to be stable, we require $\lambda_i \leq 1$. As a consequence, for:

$$\lambda_3 = -\frac{\rho_f |K|}{\rho_s |S|}\mu_i > 1$$

the partitioned scheme is unstable.

If the solid density is much larger than the fluid density (aircraft engineering, air vs. steel), the partitioned schemes work very well.

If the fluid density is similar to the solid density (biomechanics, body tissues vs. water), the the partitioned schemes do not work properly.

AITKEN RELAXATION SCHEME

In Aitken relaxation method, we use the last two iterates in order to approximate the next one.

We define the new approximated displacement to be applied on the fluid as:

$$\boldsymbol{u}^{*}_{\Gamma,\mathrm{i}} = S\left(F\left(\boldsymbol{u}_{\Gamma,i-1}\right)\right)$$

We define the residual (which is a function of $\boldsymbol{u}_{\Gamma,i-1}$) as:

$$\boldsymbol{r}^{*}_{\Gamma,i} = \boldsymbol{u}^{*}_{\Gamma,\mathrm{i}} - \boldsymbol{u}_{\Gamma,\mathrm{i}-1}$$

Now we use the secant method and we impose that $\boldsymbol{r}_{\Gamma,i+1} = 0$. We look for the $\boldsymbol{u}$ such that $\boldsymbol{r}(\boldsymbol{u}) = \boldsymbol{0}$

$$0 = \boldsymbol{r}_{\Gamma,i} + \frac{\boldsymbol{r}_{\Gamma,i} - \boldsymbol{r}_{\Gamma,i-1}}{\boldsymbol{u}_{\Gamma,\mathrm{i}} - \boldsymbol{u}_{\Gamma,\mathrm{i}-1}}\left(\boldsymbol{u}_{\Gamma,\mathrm{i}} - \boldsymbol{u}_{\Gamma,\mathrm{i}-1}\right)$$

We replace for all the epressions for $\boldsymbol{r}$, and we arrive to an expression in the form of a relaxation parameter w.

AITKEN RELAXATION SCHEME

The corrected solution $\boldsymbol{u}_{\Gamma,\mathrm{i}}$ is written as:

$$\boldsymbol{u}_{\Gamma,\mathrm{i}+1} = \boldsymbol{u}_{\Gamma,\mathrm{i}} + w(\boldsymbol{u}^{*}_{\Gamma,\mathrm{i}+1} - \boldsymbol{u}_{\Gamma,\mathrm{i}})$$

And w is defined as:

$$w = \frac{\boldsymbol{u}_{\Gamma,(i-1)} - \boldsymbol{u}_{\Gamma,\mathrm{i}}}{\boldsymbol{u}_{\Gamma,(i-1)} - \boldsymbol{u}_{\Gamma,\mathrm{i}} + \boldsymbol{u}^{*}_{\Gamma,(i+1)} - \boldsymbol{u}^{*}_{\Gamma,\mathrm{i}}}$$

This helps to alleviate the added mass effect.

Other possibilities are:

- Steepest descent methods.
- Robin-Robin boundary conditions.