UPC

Computational Mechanic Tolls

PDE Toolbox

Assignment 1

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by

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We have to solve the following problem:

$$u_t - \Delta u = f \quad in \;\; \Omega = [0,1]^2 \tag{1}$$

where $f\,(x, y, t) = -3e^{-3t}$ using the MATLAB PDE Toolbox.

Moreover, we have as initial condition at $t = 0$: $u(x, y, t = 0) = x2 + xy - y2 + 1$ and also the following boundary conditions:
$u_n(x = 0, y, t) = -y$, $u_n(x = 1, y, t) = 2 + y$, $u(x, y = 0, t) = x^2 + e^{-3t}$, $u_n(x, y = 1, t) = x - 2$ where $u_n = \delta u/\delta n$.

The analytical solution of this problem is:

$$u(x, y, t) = x^2 + xy - y^2 + e^{-3t} \tag{2}$$

and knowing it we can calculate the maximum error between the analytical solution and the numerican one, found with PDE tollbox.

- Now we solve the parabolic problem considering $t_{end} = 10$. We start from a coarse mesh and we refine it 4 times in order to have a preciser solution and check how it's the convergence slope.
  For the first coarse mesh we have the following error:

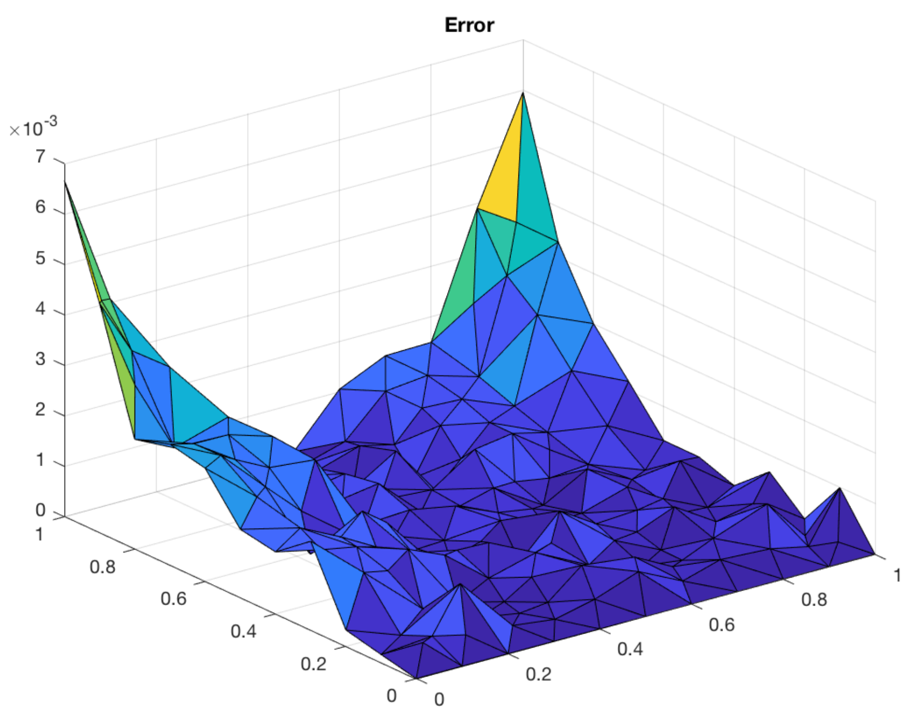

Figure 1: plot of the error for a h=0.104 and final time t=10

that becomes better refining the mesh:

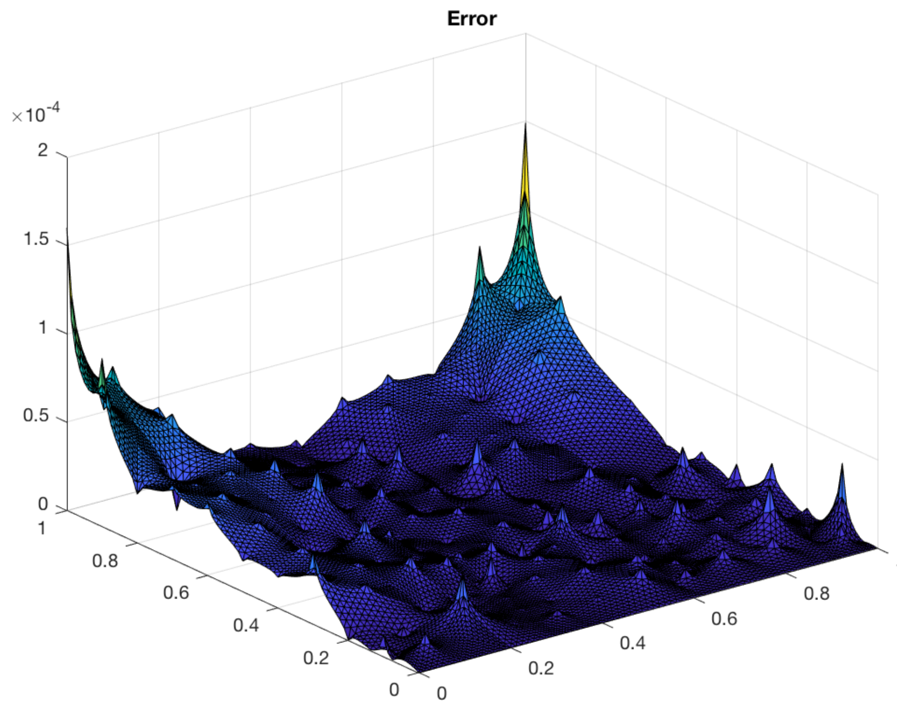

Figure 2: plot of the error for a h=0.0098 and final time t=10

As we can see there are points (the corner in (0,1) and (1,1)), where the error remains bigger also after the refinement.
This is due to the high gradient of the solution in those zones. Infact the solution has to respect the different boundary conditions for each side of the square and the corner is the point where the two different conditions are the nearest; so there is the biggest error.
But, as we can see in a qualitative way from figure(1) and figure(2) the error tends to decrease in general.
In a quantitative way, we can be more precise and see how the error goes to zero. The result is coerent with the theoretical one, infact we have found a slope of h^2.

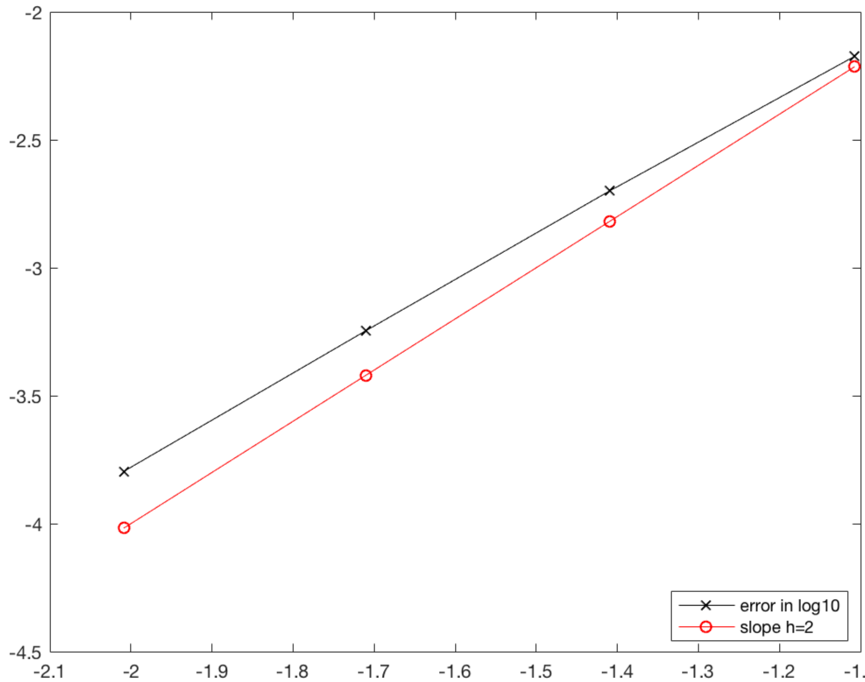

Figure 3: plot of the log of the error with respect to the log of the h comparing with a line of slope 2

As we can see the theoretical convergence is satisfied.
Now, knowing that the max error tends to zero we know that our numerical solution is correct and there aren't any problems in the resolution (as numerical oscillation). Below there are the solution seen in different configurations.

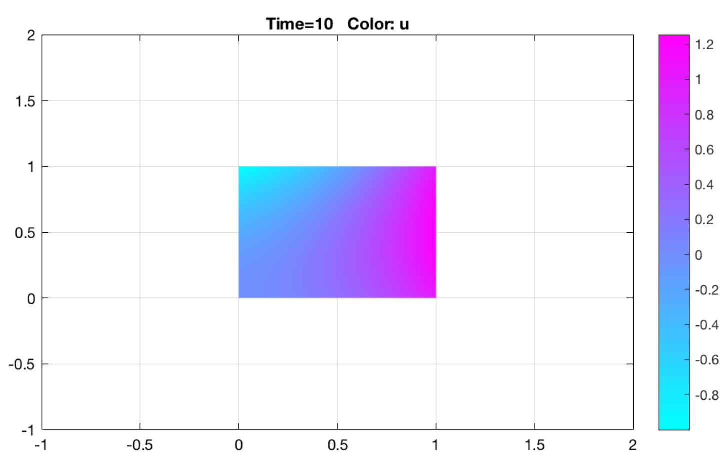

Figure 4: plot of the numerical solution in 2D

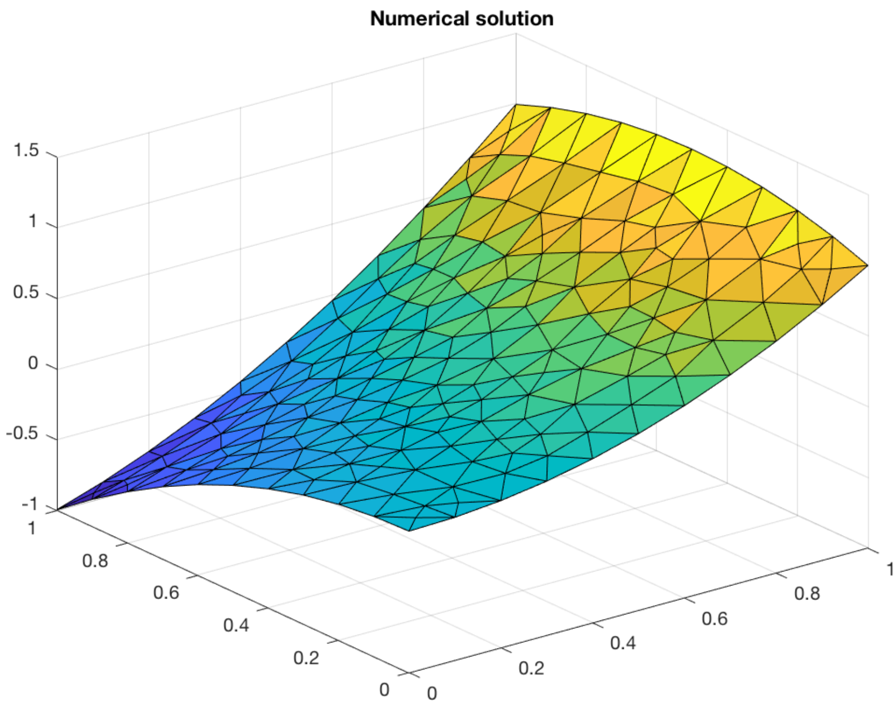

Figure 5: plot of the numerical solution in 3D

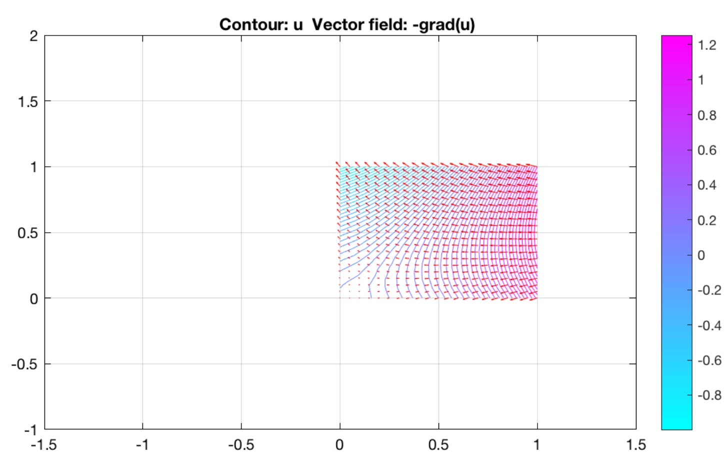

Figure 6: plot of the numerical solution in 2D with the level lines and the arrows of the direction of the heat

As we can see in (1,1) there is the biggest gradient of the the solution, as we had already checked in the previous figure of the error.
At the end, we can comparing the numerical result to the analytical one and check that they are consistent.

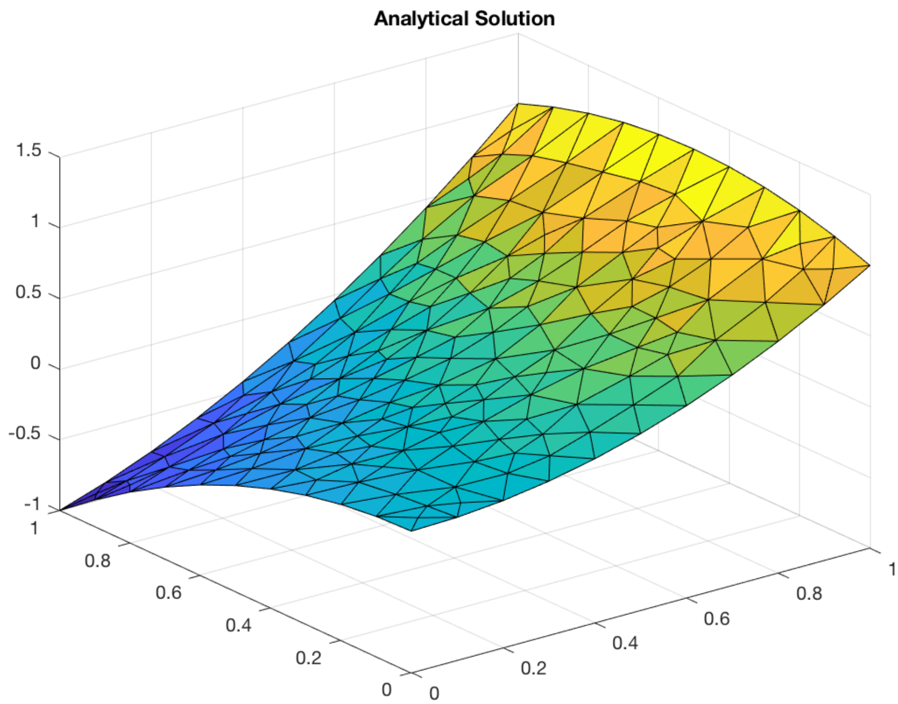

Figure 7: plot of the analytical solution in 3D

- The second part is control how change the solution in time.
  Seeing the boundary conditions, the initial condition and the source term f, we can note as the time appears always in this form: e^{-3t}. It means that increasing t the contributes of the time to the problem becomes smaller and smaller till to disappear after $t = 5$ since it's over the precision of calculus of a normal computer (that is usually of e^{-16}).
  So the time gives a contribute to the heat (by the source term) only in the first moments; for example for t=1 $e^{-3t} = 0.04978706836$, but already for t=2 is $e^{-3t} = 0.00247875217$. So we expect that the solution at $t = 1$ is more or less the same of that after $t = 10$ (maybe only a little smaller) since it would be already stabilized. Infact:

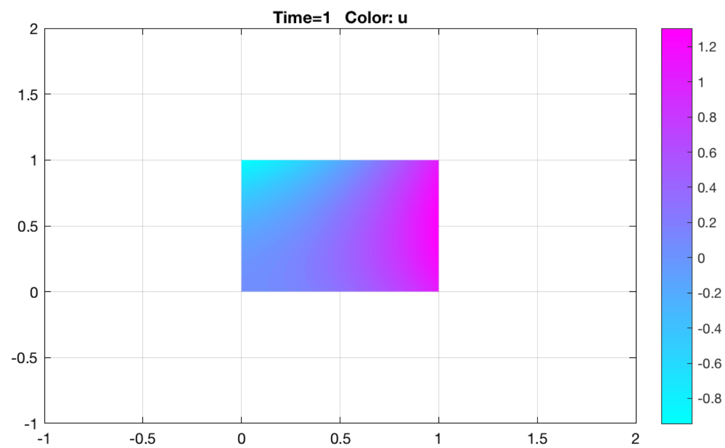

Figure 8: plot of the numerical solution in 2D

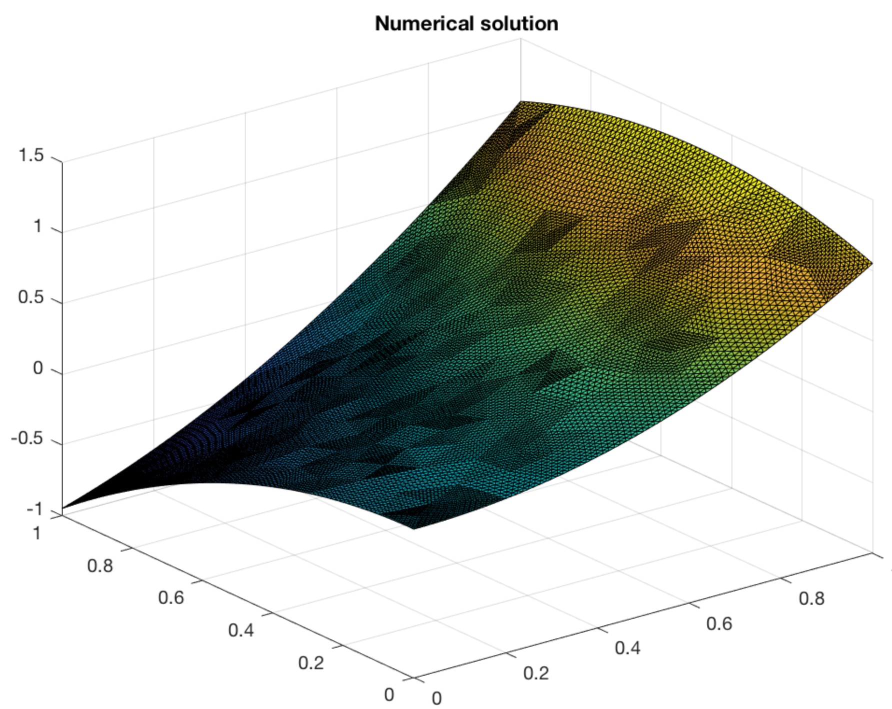

Figure 9: plot of the numerical solution in 3D

The error here it's a little bigger than the error at time t=10 since the solution is not so smooth as the one at t=10. This is due to the work of the diffusive term that makes the solution always smoother, more the time increase.
Indeed we have the following result about the error:

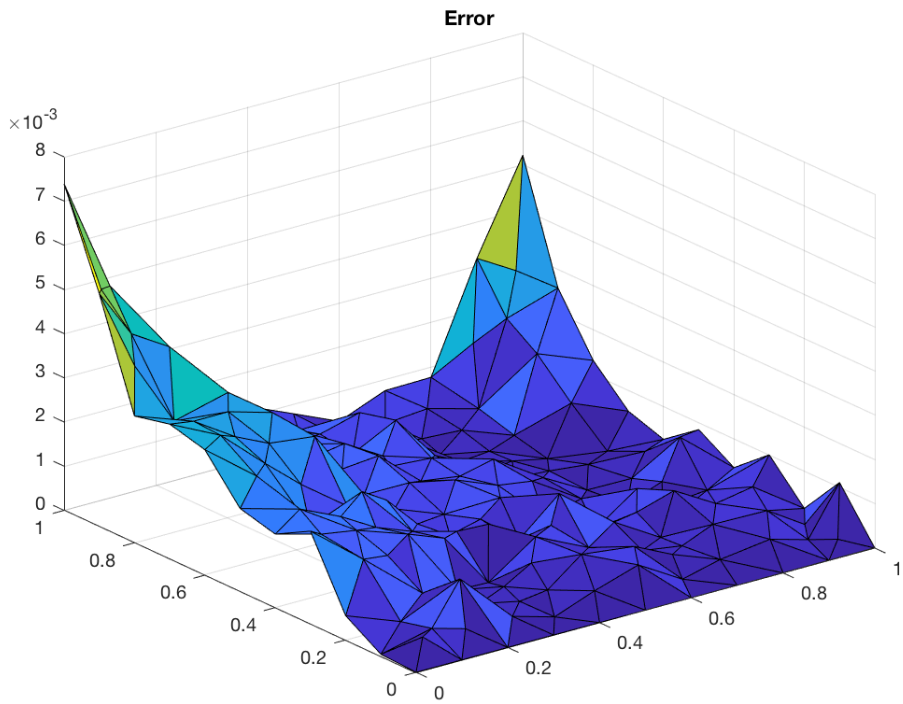

Figure 10: error in 3D a t=1 with the first coarse mesh (h=0.104)

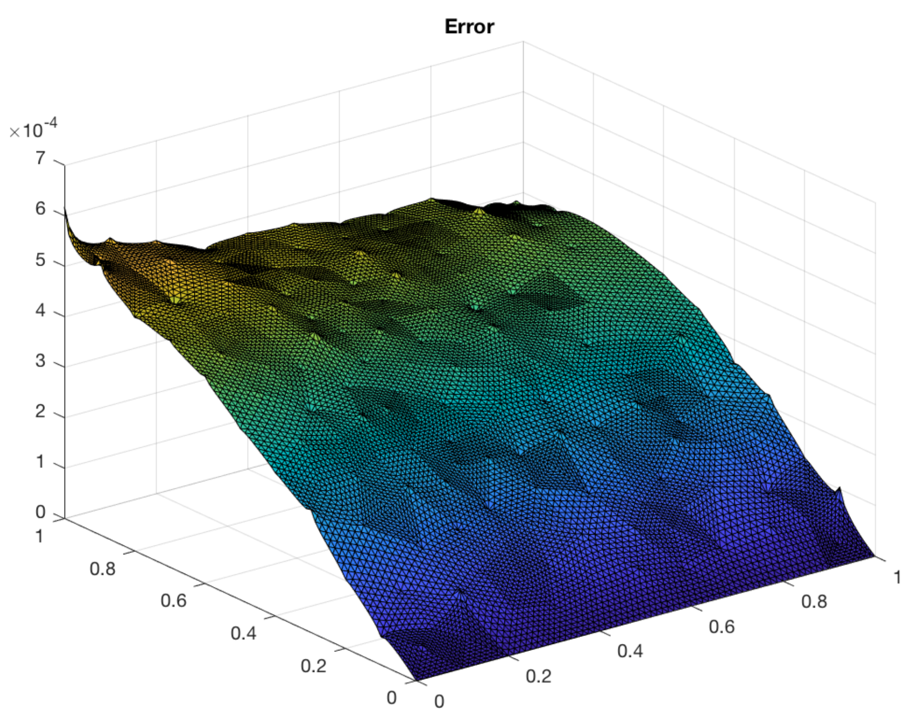

Figure 11: error in 3D a t=1 with h=0.009

- We have also found the solution at time t=50. As we expected, the problem it is stabilized around a constant solution after t=1.5 (more or less). So the solution at t=50 will be the same of the one at t=10 (maybe only a bit more smoother). Infact we have found all these concepts in the figures below:

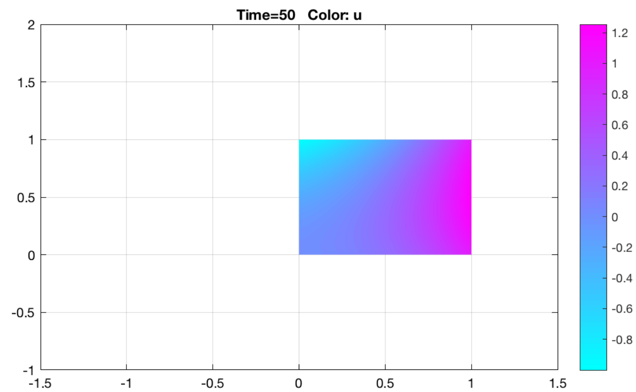

Figure 12: numerical solution in 2D at time t=50

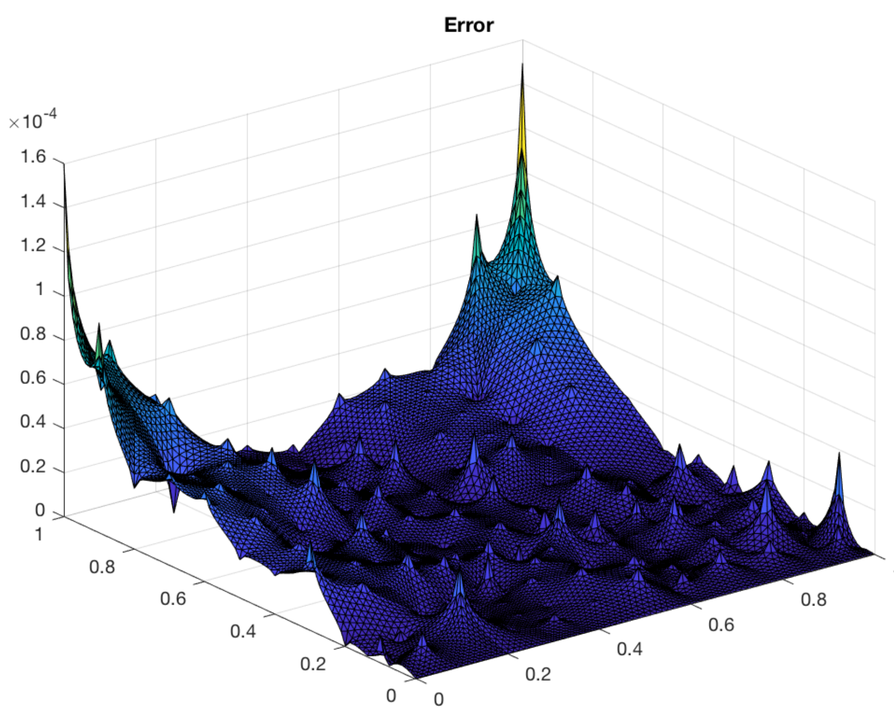

Figure 13: error at t=50 for a fine mesh

We have found a slope a little bigger than the one a t=10:
indeed while the first has a slope = 1.8407 , this one has 1.8418. It's not a big difference, but it's important to see the work of the diffusion.

Obviusly, this is not an efficient way to solve the problem since it's much expensive also for a very easy configuration as a square. To solve this problem in an equivalent way we can follow two different paths.
The first and more precise is to calculate the solution till a smaller t, for example t=10 or also less. The difference where will be very little.
Another way could be consider the problem as a stationary one (elliptic) and not more parabolic. In this way we reduce a lot the computational cost (also respect to the one with t troncated ad $t = 10$). So trasforming it to an elliptic problem we have to delete all the presences of t in the equation/conditions. If we subsitute $t = 0$ in all these equation (so it's equivalente to erase alle the expential) we find a solution consistent with the parabolic problem but smaller than the parabolic after t=1 (and so also t=10 and so on).

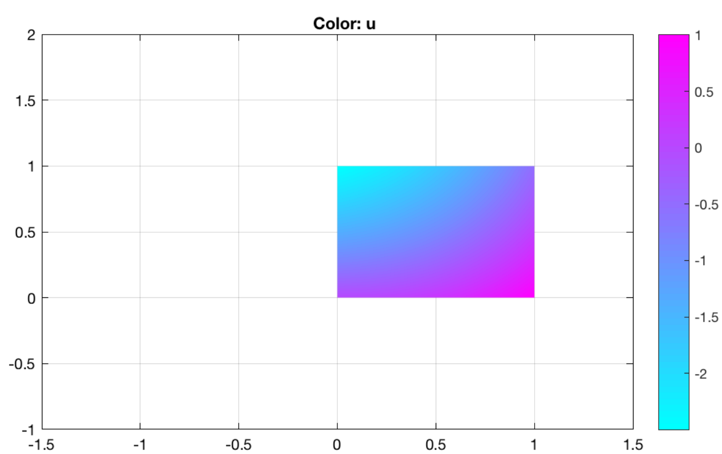

Figure 14: Solution 2D with t=0

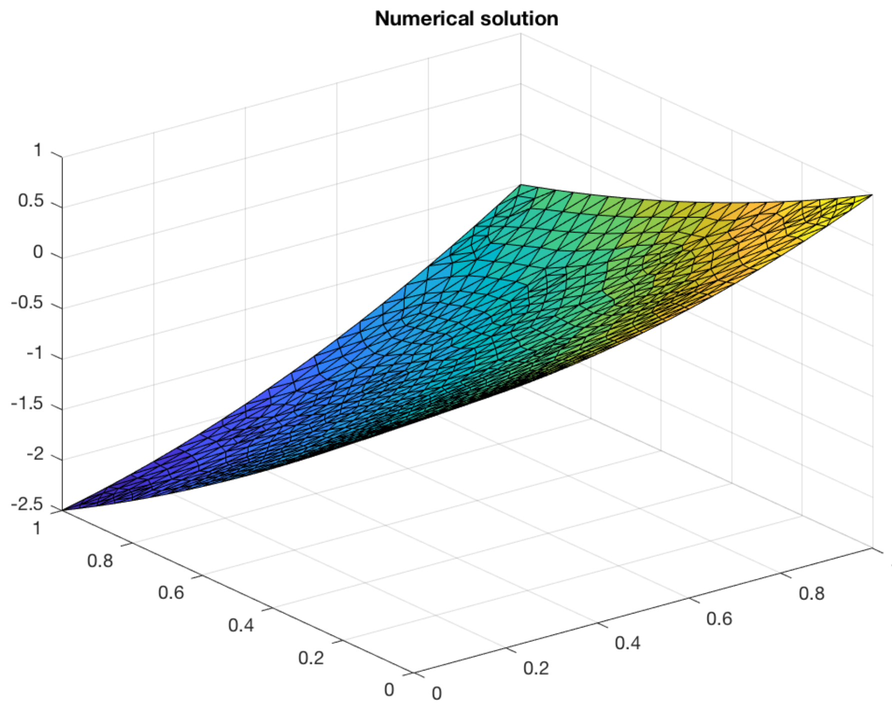

Figure 15: Solution 3D with t=0

To compensate for this time-part deleted, for example we could not eliminate the exponential term but consider the t as a constant fixed. I have tried with t=3.33 (i.e I have all e^{-10}) and the results are very similar to the one found in the parabolic case, as the figures below show

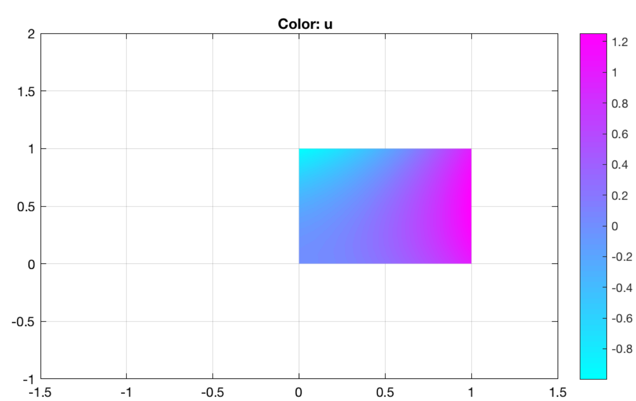

Figure 16: numerical solution of the elliptic problem with t=3.333 fixed

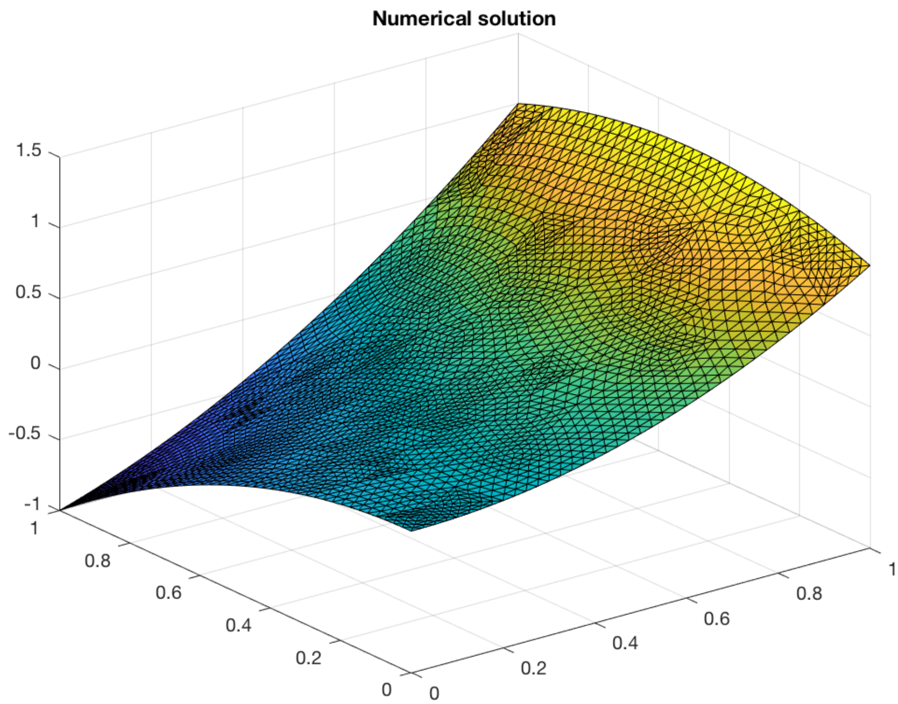

Figure 17: numerical solution of the elliptic problem for t=1.666 fixed

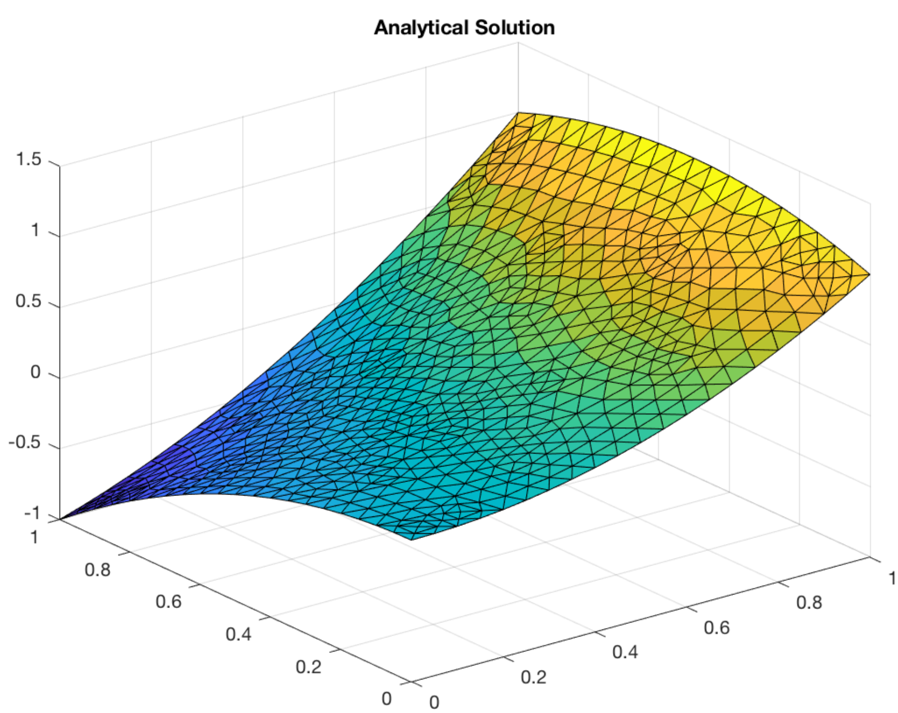

Figure 18: analytical solution of the parabolic problem for t=10

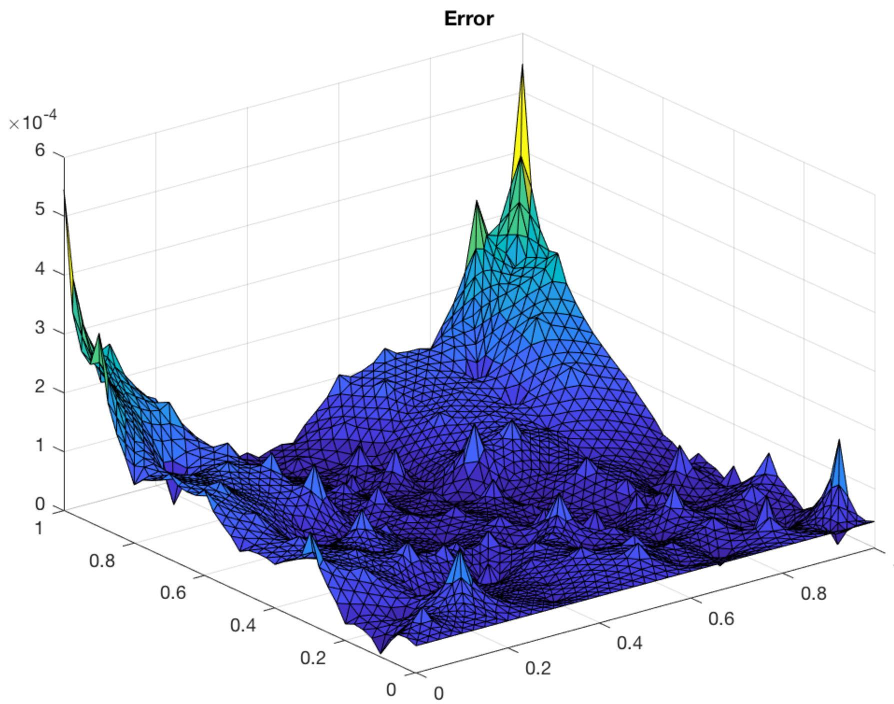

Figure 19: error of the elliptic problem with t=3.333 fixed. the max error between this and the parabolic one is equal to $5.4747e - 04$ using an h=0.0195

I think that it is possible to improve more the research of an equivalent elliptic problem finding the right constant that, substitued, gives us the more similar solution of our parabolic problem.