Universitat Politècnica de Catalunya
Numerical Methods in Engineering
Computational Mechanics Tools

Assignment 3

PDE toolbox

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Contents

Abstract

In this document I report my results on following the instructions proposed in assignment 3. These entail the solving of a time-dependent value boundary problem with the PDE toolbox for Matlab with different solver options. The problem solved is elliptic.

Problem statement

We are asked to solve the following problem:

$$\left.\begin{aligned} &\tfrac{\partial u}{\partial t} - \Delta u = -e^{-3t} && \text{in } \Omega \\ &\tfrac{\partial u}{\partial n} = -y && \text{on } \Gamma_{left} \\ &\tfrac{\partial u}{\partial n} = 2 + y && \text{on } \Gamma_{right} \\ &u = x^2 + e^{-3t} && \text{on } \Gamma_{bottom} \\ &\tfrac{\partial u}{\partial n} = x - 2 && \text{on } \Gamma_{top} \\ &u = x^2 + xy - y^2 + 1 && \text{at } t = 0 \end{aligned}\right\} \tag{1}$$

where Ω is a unit square $[0, 1] \times [0, 1]$ with border Γ. We are also given the analytical solution to the problem:

$$u(x, y, t) = x^2 + xy - y^2 + e^{-3t} \tag{2}$$

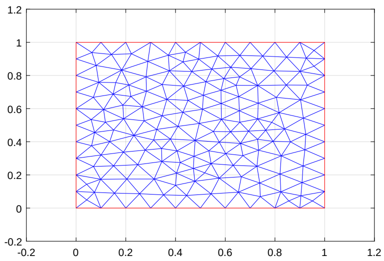
With this we have everything we need to start solving the problem.

1 Refining the mesh

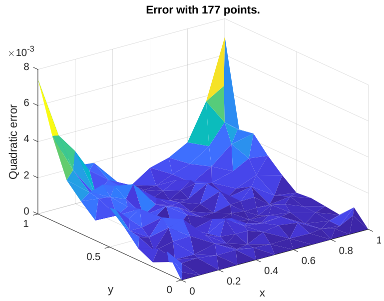
We are asked to solve for $t \in [0, 10]$ with four increasingly thinner meshes. We'll solve it with time-steps of $0.5s$. The results can be found in figure 1. The error is calculated at the end of the time interval since that's when it should be maximum. The error and mean error have been computed as:

$$\varepsilon(x, y) = ||u_a(x, y) - u_{\text{num}}(x, y)||$$
$$\bar{\varepsilon} = \frac{1}{n} \sum_{i=1}^{n} \varepsilon(x_i, y_i)$$

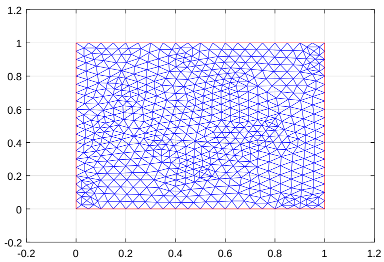
where n is the number of nodes, u_a is the analytical solution and u_{num} is the numerical approximation.



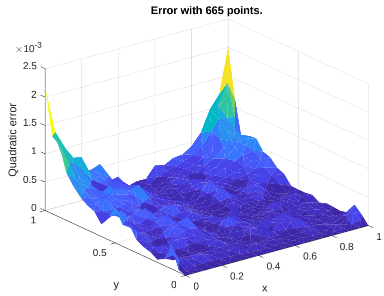
(a) Mesh with 328 elements



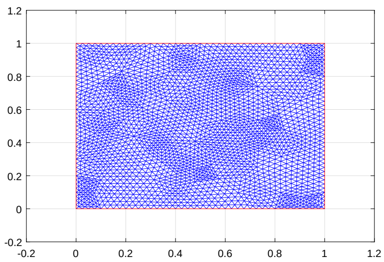
(b) Mean error of 8.821×10^{-4}



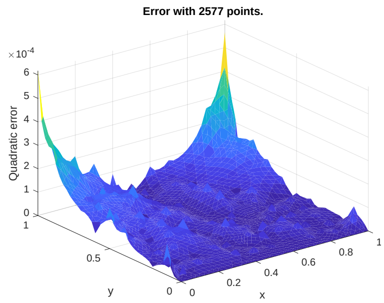
(a) Mesh with 1312 elements



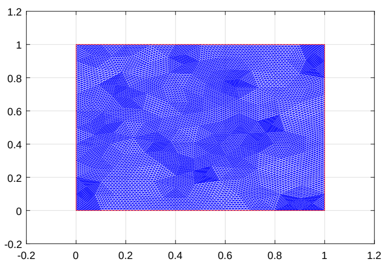
(b) Mean error of 2.064×10^{-4}



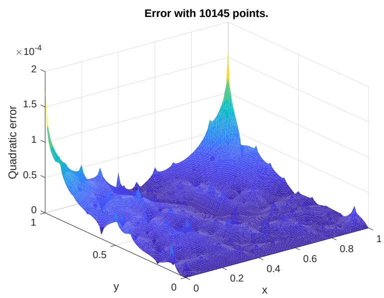
(a) Mesh with 5248 elements



(b) Mean error of 5.000×10^{-5}



(a) Mesh with 20992 elements



(b) Mean error of 1.233×10^{-5}

Figure 1: Meshes and error maps evaluated at $t = 10$. Zoom in to see the thinner meshes.

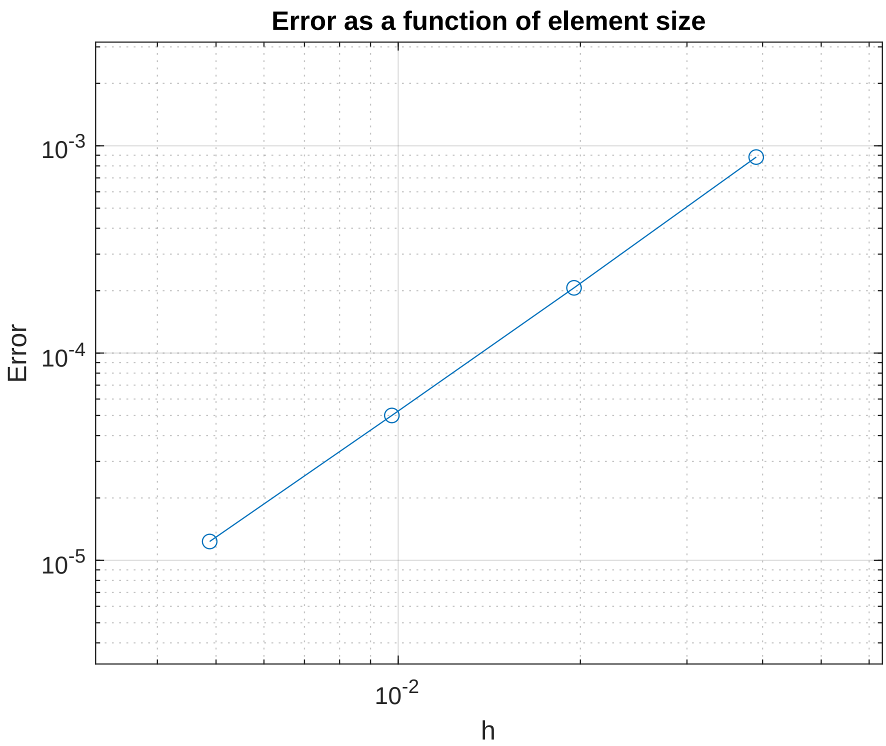


Figure 2: Error as a function of element size

If we plot the error versus. the mean element size we obtain the plot in figure 2. The mean size for the elements is computed as:

$$h = \sqrt{\frac{A}{2n_e}} \tag{3}$$

The slope of the line is 2.14, different from the theoretical 1.8. This implies a possible mistake in the calculation of the error.

2 Changing the time interval

In this section we are asked to solve the same system but for different end-times and compare the results. To do so I used the thinnest mesh from the previous section and obtained the following results:

t_{end}	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0
Mean error	0.9513	1.3861	1.8339	2.2823	2.7307	3.1791	3.6275	4.0759	4.5244	4.9728

Table 1: Error as a function of time.

As can be seen, as time goes on the error accumulates. Hence, longer time intervals will cause larger discrepancies between analytical and numerical solutions.

3 Large values of t

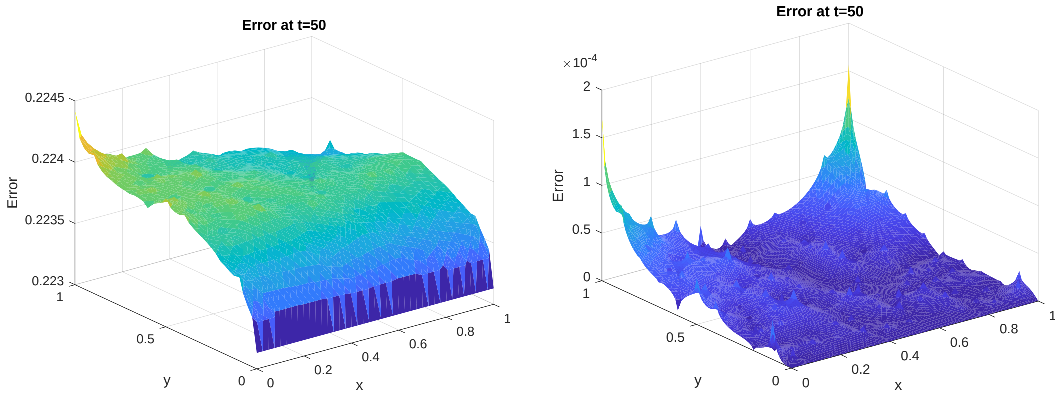
In this final section we are asked to provide a solution for $t = 50$ in a more accurate manner than the method we've been using before. If we pay attention to equation 2 we will notice that:

$$\lim_{t\to\infty} u(x, y, t) = x^2 + xy - y^2 \tag{4}$$

That is, this function tends towards a steady state. The solution, then, becomes time-independent for large values of t. The term that disappears is e^{-3t}, and for $t = 50$ it becomes 7.1751×10^{-66}, i.e. negligible. Needless to say, $\partial u/\partial t$ becomes negligible as well. Hence, we can use a time independent solver to solve the following problem:

$$\left.\begin{array}{ll} \Delta u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = -y & \text{on } \Gamma_{left} \\ \frac{\partial u}{\partial n} = 2 + y & \text{on } \Gamma_{right} \\ u = x^2 & \text{on } \Gamma_{bottom} \\ \frac{\partial u}{\partial n} = x - 2 & \text{on } \Gamma_{top} \end{array}\right\} \tag{5}$$

The results can be seen in figure 3.



(a) Iterative. Mean error of 0.235.

(b) Steady state. Mean error of 2.213×10^{-5}.

Figure 3: Results evaluated at $t = 50$