1D FEM Problem - Obtaining Stiffness matrix

Consider the differential equation that models a physical problem,

$$-u'' = f, \quad \text{for } x \in [a,b]$$

with boundary conditions, $u(a) = u(b) = 0$
Multiply by a weight/test function v and integrate in the whole domain, we get,

$$\int_a^b -vu''dx = \int_a^b vf\,dx$$

Using Integration by parts on the LHS, we get,

$$\int_a^b v'u'dx - [v'u]_a^b = \int_a^b vf\,dx$$

Now, after imposing the boundary conditions, we get the weak form of the problem as,

$$\boxed{\int_a^b v'u'dx = \int_a^b vf\,dx}$$

The FE approximate solution is obtained by replacing u with a trial solution $\overline{u}$ that is built using a finite element mesh where we associate a shape function to each node of the FE discretisation.

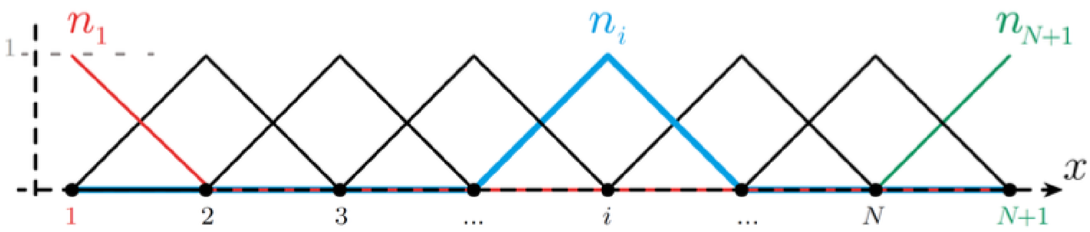

Figure 1: Shape functions

The following approximation to u is used in the numerical solution of the weak form,

$$u(x) \approx \overline{u}(x) = \sum_{i=1}^{N+1} u_j\, N_j(x)$$

Substituting the approximation and $v = N_i$, we get,

$$\int_a^b N_i' \left(\sum_{i=1}^{N+1} u_j\, N_j'(x) \right) dx = \int_a^b N_i\, f\,dx$$

$$\sum_{i=1}^{N+1} \left(\int_a^b N_i'\, N_j'\, dx \right) u_j = \int_a^b N_i\, f\,dx$$

The above equation can be written in the form,

$$\boxed{K\ u\ =\ f}$$

where,

$$K_{ij} = \int_a^b N_i'\ N_j'\ dx, \qquad f_i\ =\ \int_a^b N_i\ f\ dx$$

Now, we do the element-by-element assembly For a 2-noded (linear) element,

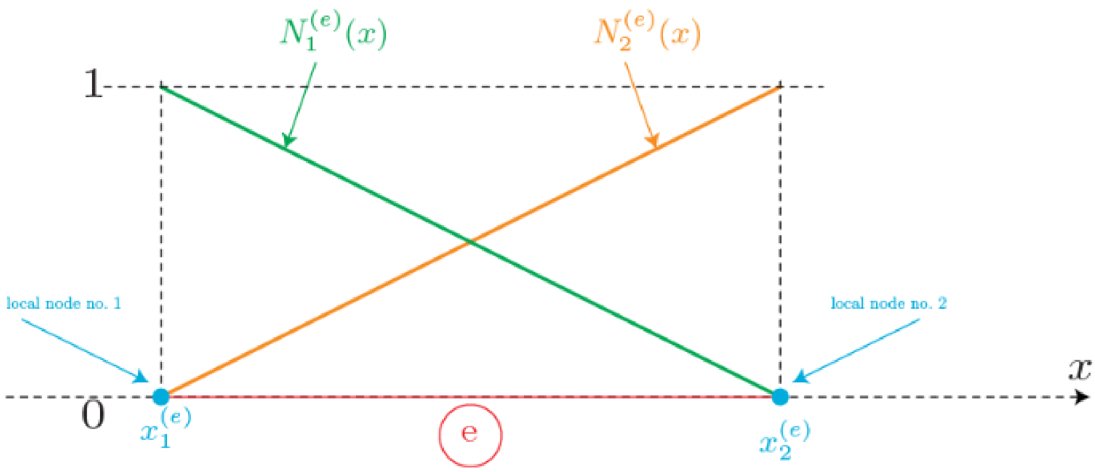

Figure 2: Element shape function

$$N_1^{(e)} = \frac{x_2^{(e)} - x}{x_2^{(e)} - x_1^{(e)}} = \frac{x_2^{(e)} - x}{l^{(e)}}$$

$$N_2^{(e)} = \frac{x - x_1^{(e)}}{x_2^{(e)} - x_1^{(e)}} = \frac{x - x_1^{(e)}}{l^{(e)}}$$

$$\frac{dN_1^{(e)}}{dx} = \frac{-1}{x_2^{(e)} - x_1^{(e)}} = \frac{-1}{l^{(e)}}$$

$$\frac{dN_2^{(e)}}{dx} = \frac{1}{x_2^{(e)} - x_1^{(e)}} = \frac{1}{l^{(e)}}$$

The global stiffness matrix is then constructed by assembling the element contributions. For a linear element the global matrix is assembled as,

$$K = \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & & & & & \\ K_{21}^{(1)} & K_{22}^{(1)} + K_{11}^{(2)} & K_{12}^{(2)} & & & & \\ & K_{21}^{(2)} & K_{22}^{(2)} + K_{11}^{(3)} & K_{12}^{(3)} & & & \\ & & K_{21}^{(3)} & K_{22}^{(3)} + K_{11}^{(4)} & & & \\ & & & & \dots & & \\ & & & & & K_{22}^{(n-1)} + K_{11}^{(n)} & K_{12}^{(n)} \\ & & & & & K_{21}^{(n)} & K_{22}^{(n)} \end{bmatrix}$$

where, n is the number of elements. We know that, for each element,

$$K_{ij}^{(e)} = \int_{x_1^{(e)}}^{x_2^{(e)}} N_i' N_j' \, dx = \begin{cases} \frac{1}{l^{(e)}} & \text{if } i = j \\ -\frac{1}{l^{(e)}} & \text{if } i \neq j \end{cases}$$

Hence, we get the global stiffness matrix as,

$$K = \begin{bmatrix} \frac{1}{l^{(1)}} & -\frac{1}{l^{(1)}} & & & & & \\ -\frac{1}{l^{(1)}} & \frac{1}{l^{(1)}} + \frac{1}{l^{(2)}} & -\frac{1}{l^{(2)}} & & & & \\ & -\frac{1}{l^{(2)}} & \frac{1}{l^{(2)}} + \frac{1}{l^{(3)}} & -\frac{1}{l^{(3)}} & & & \\ & & -\frac{1}{l^{(3)}} & \frac{1}{l^{(3)}} + \frac{1}{l^{(4)}} & & & \\ & & & & \dots & & \\ & & & & & \frac{1}{l^{(n-1)}} + \frac{1}{l^{(n)}} & -\frac{1}{l^{(n)}} \\ & & & & & -\frac{1}{l^{(n)}} & \frac{1}{l^{(n)}} \end{bmatrix}$$

Therefore, if the number of elements $n = 6$ and the length of the elements are given as, $l^{(1)}$, $l^{(2)}$, $l^{(3)}$, $l^{(4)}$, $l^{(5)}$ and $l^{(6)}$ as shown in the figure below

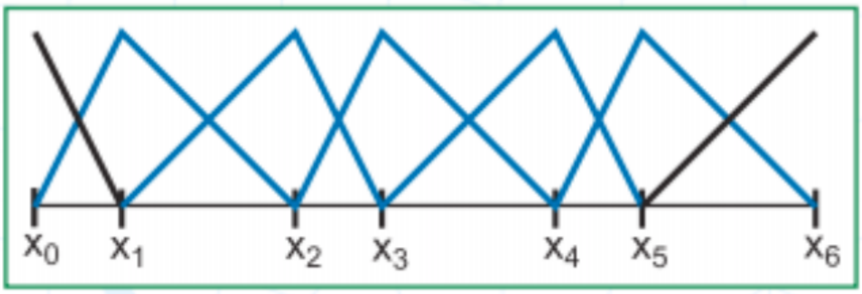

Figure 3: 1D domain - number of elements = 6

we get the following global stiffness matrix,

$$K = \begin{bmatrix} \frac{1}{l^{(1)}} & -\frac{1}{l^{(1)}} & & & & & \\ -\frac{1}{l^{(1)}} & \frac{1}{l^{(1)}} + \frac{1}{l^{(2)}} & -\frac{1}{l^{(2)}} & & & & \\ & -\frac{1}{l^{(2)}} & \frac{1}{l^{(2)}} + \frac{1}{l^{(3)}} & -\frac{1}{l^{(3)}} & & & \\ & & -\frac{1}{l^{(3)}} & \frac{1}{l^{(3)}} + \frac{1}{l^{(4)}} & -\frac{1}{l^{(4)}} & & \\ & & & -\frac{1}{l^{(4)}} & \frac{1}{l^{(4)}} + \frac{1}{l^{(5)}} & -\frac{1}{l^{(5)}} & \\ & & & & -\frac{1}{l^{(5)}} & \frac{1}{l^{(5)}} + \frac{1}{l^{(6)}} & -\frac{1}{l^{(6)}} \\ & & & & & -\frac{1}{l^{(6)}} & \frac{1}{l^{(6)}} \end{bmatrix}$$

Now, with the given dirichlet boundary conditions, $u(a) = u(b) = 0$
We remove the corresponding rows and column from the matrix and the stiffness matrix is reduced to,

$$K = \begin{bmatrix} \frac{1}{l^{(1)}} + \frac{1}{l^{(2)}} & -\frac{1}{l^{(2)}} & & & \\ -\frac{1}{l^{(2)}} & \frac{1}{l^{(2)}} + \frac{1}{l^{(3)}} & -\frac{1}{l^{(3)}} & & \\ & -\frac{1}{l^{(3)}} & \frac{1}{l^{(3)}} + \frac{1}{l^{(4)}} & -\frac{1}{l^{(4)}} & \\ & & -\frac{1}{l^{(4)}} & \frac{1}{l^{(4)}} + \frac{1}{l^{(5)}} & -\frac{1}{l^{(5)}} \\ & & & -\frac{1}{l^{(5)}} & \frac{1}{l^{(5)}} + \frac{1}{l^{(6)}} \end{bmatrix}$$