PDE-Toolbox
Assignment 3

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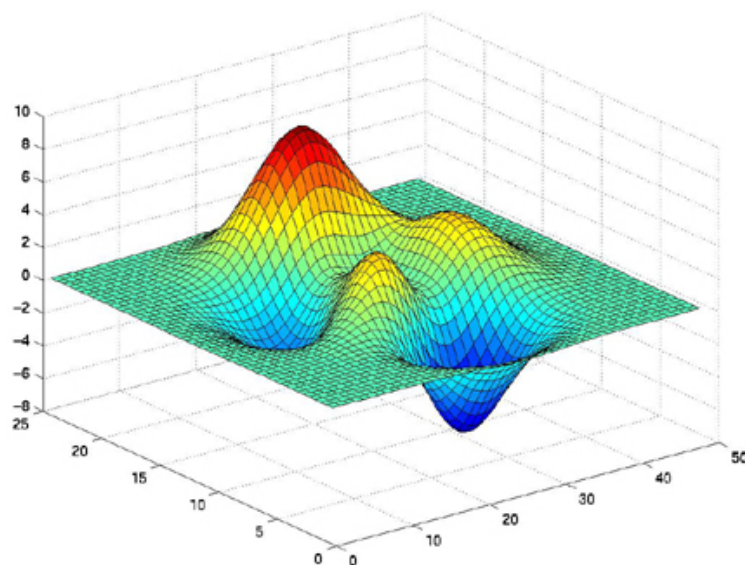
Contents

1   Introduction

Partial Differential Equation Toolbox provides functions for solving structural mechanics, heat transfer, and general partial differential equations (PDEs) using finite element analysis.

In this assignment, we are asked to compute a given parabolic differential equation with the following conditions:

$$u_t - \Delta u = f \quad in \ \Omega = [0,1]^2, \tag{1}$$

where the source term is given by,

$$f(x,y,t) = -3e^{-3t} \tag{2}$$

Considering an initial condition at t=0:

$$u(x,y,t=0) = x^2 + xy - y^2 + 1 \tag{3}$$

and the following boundary conditions

$$u_n(x=0,y,t) = -y \quad Neumann \tag{4}$$

$$u_n(x=1,y,t) = 2+y \quad Neumann \tag{5}$$

$$u(x,y=0,t) = x^2 + e^{-3t} \quad Dirichlet \tag{6}$$

$$u_n(x,y=1,t) = x-2 \quad Neumann \tag{7}$$

where $u_n = \frac{\partial u}{\partial n}$.

The analytical solution of this problem is given by the following expression,

$$u(x,y,t) = x^2 + xy - y^2 + e^{-3t} \tag{8}$$

Where the numerical solution will be computed through the Matlab PDE toolbox.

2   Results

Considering a $t_{end} = 10$, we are asked to solve the problem, and refine the initial mesh up to 4 times. Verifying that the theoretical convergence order holds.

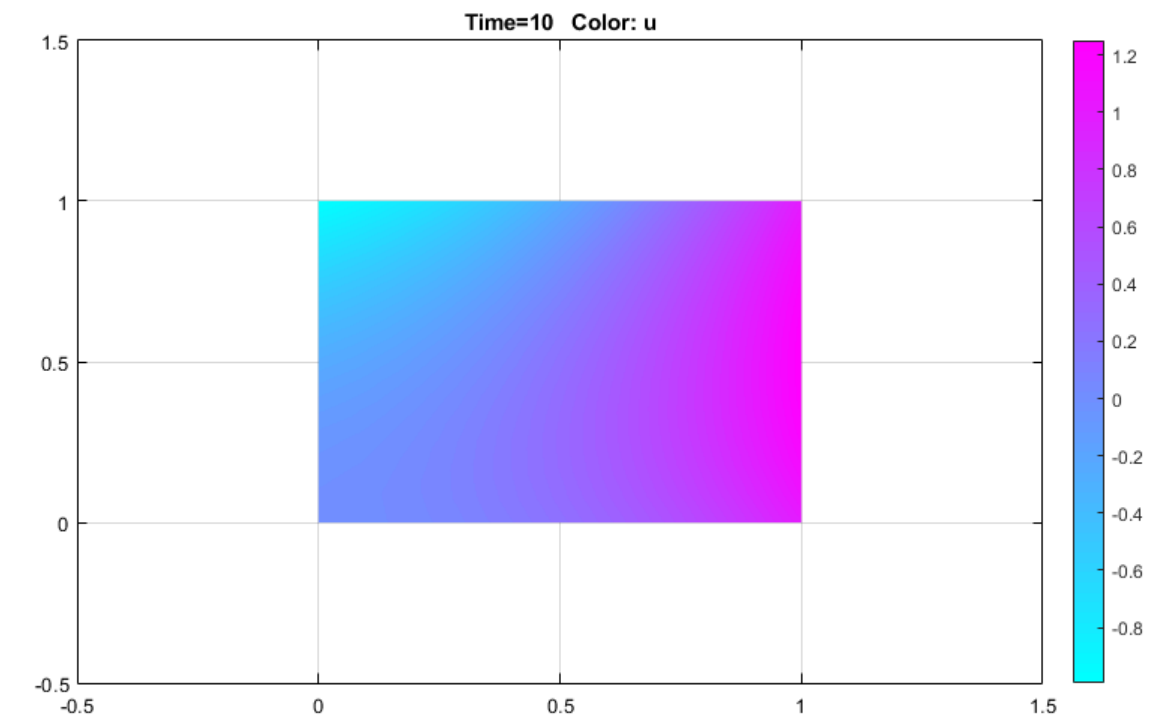The mesh size h for the meshes was computed considering the mean size of all the elements contained in Ω by:
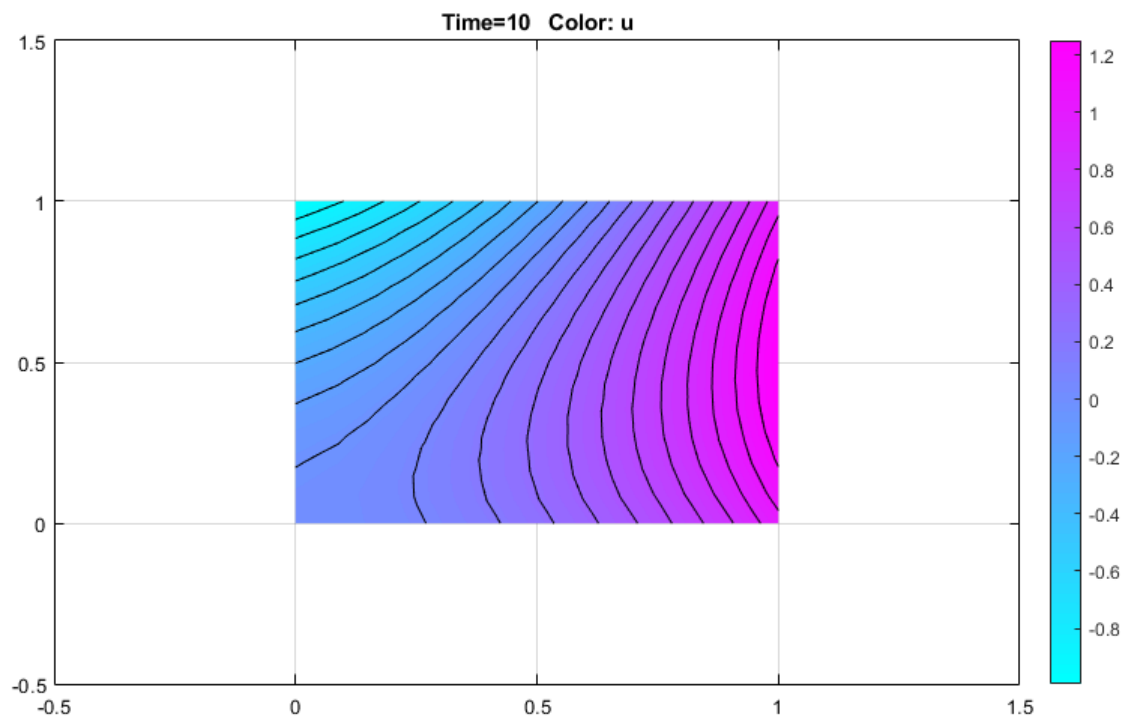
$$h = \sqrt{\frac{2A}{n}}$$

Where:
A: total area of Ω
n: number of elements in Ω
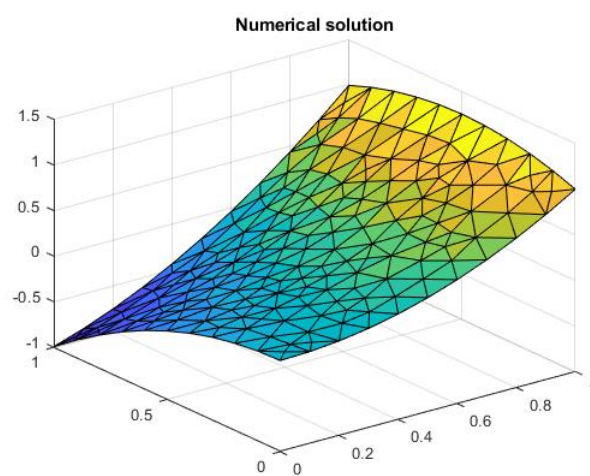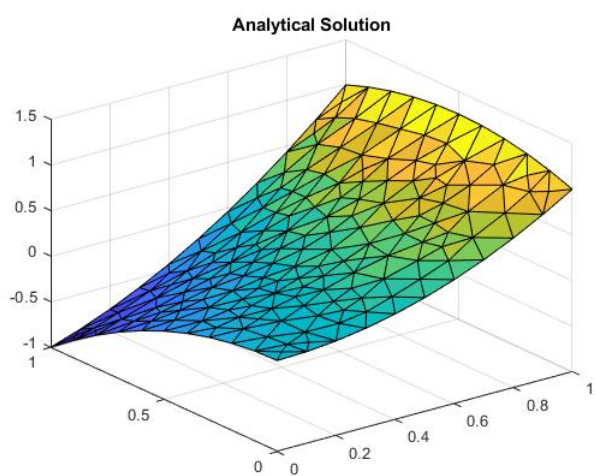
Once it is computed, the PDE toolbox shows the next diagram:

or

2.1 Initial mesh
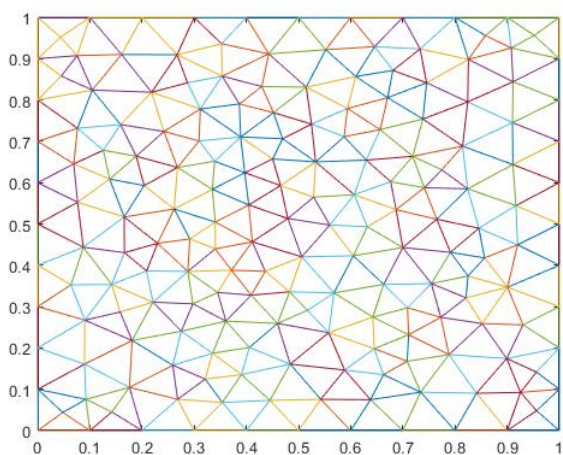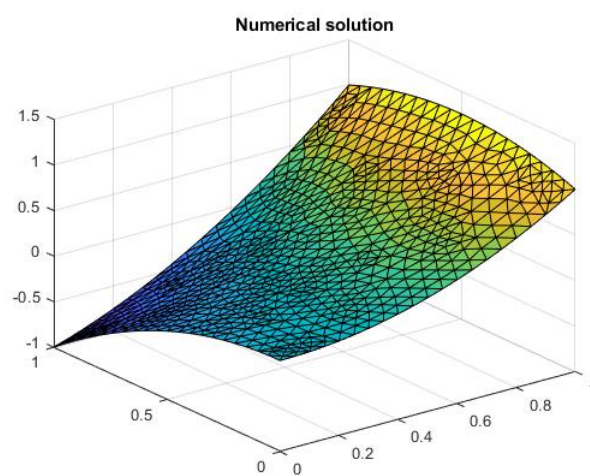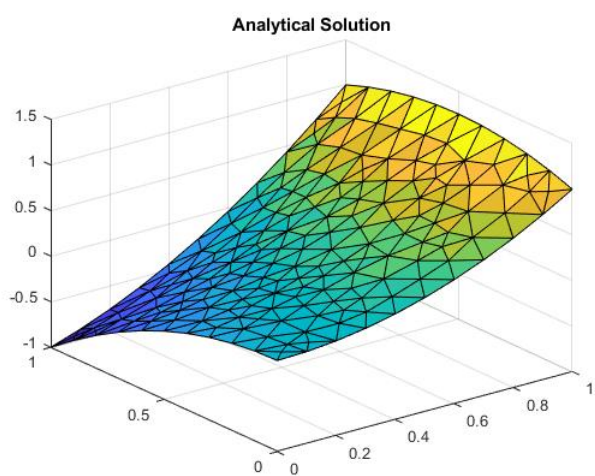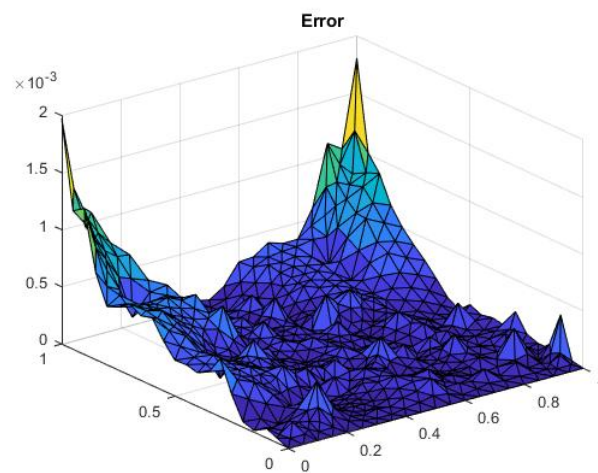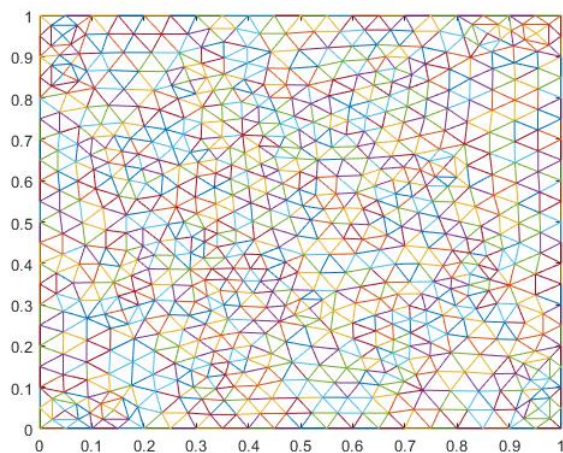
After computing the numerical and analytical solution for the 1st mesh, we obtained the following graphs:
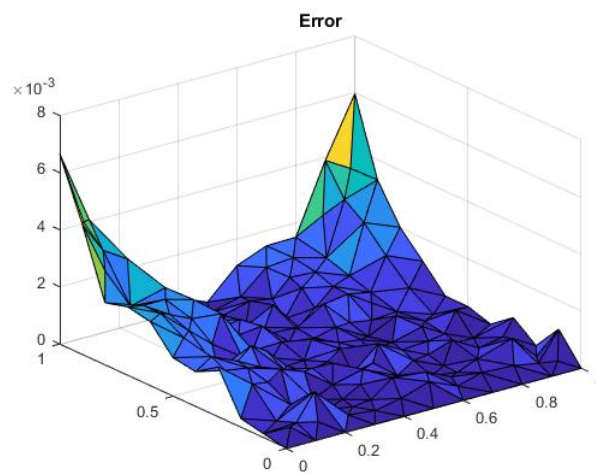
Obtaining a maximum error of 0.006662135271408. With $h = 0.078086880944303$.
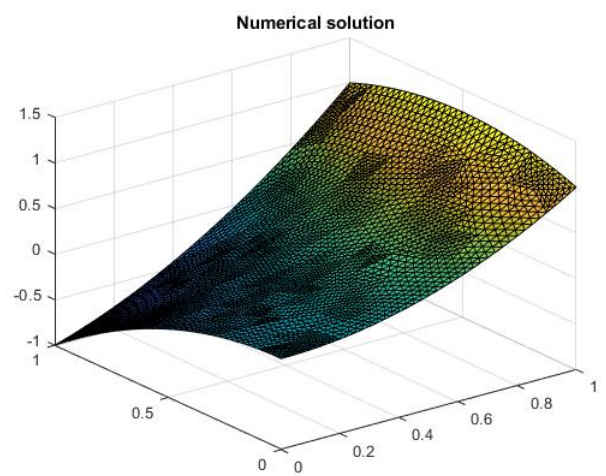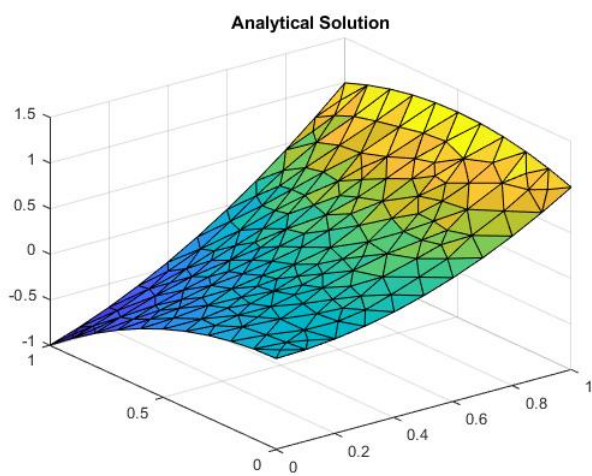
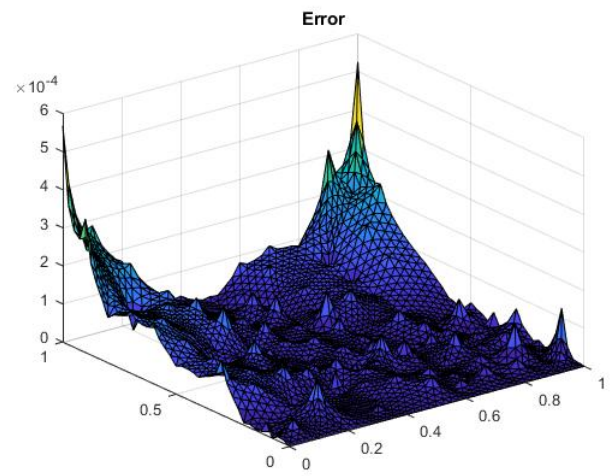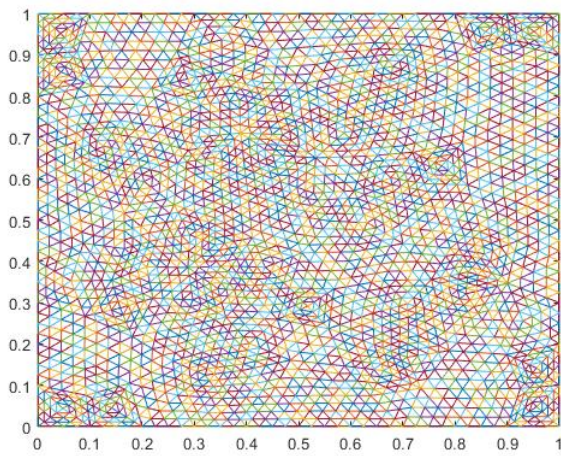## 2.2   Refined mesh 1

If we refine our mesh once, we get the following results:

Obtaining a maximum error of 0.001973332296642. With $h = 0.039043440472152$.
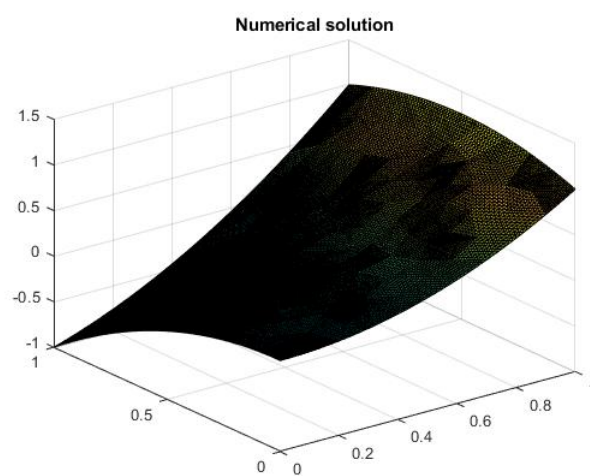
2.3   Refined mesh 2

If we refine our mesh twice, we get the following results:

Obtaining a maximum error of 5.678745361570314e-04. With $h = 0.019521720236076$.

2.4   Refined mesh 3

If we refine our mesh three times, we get the following results:
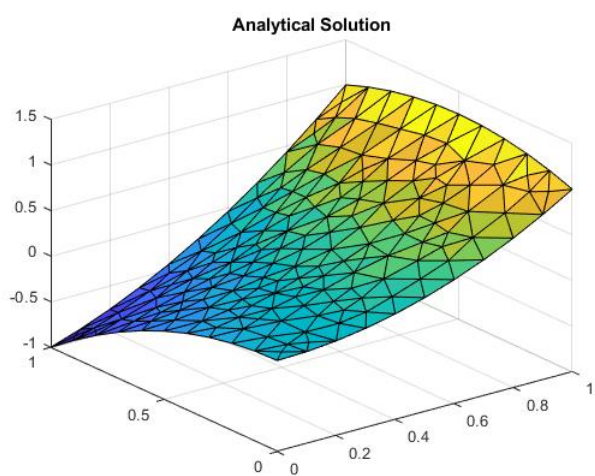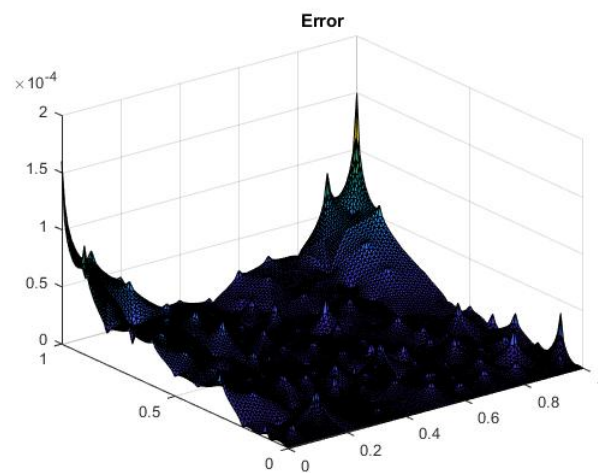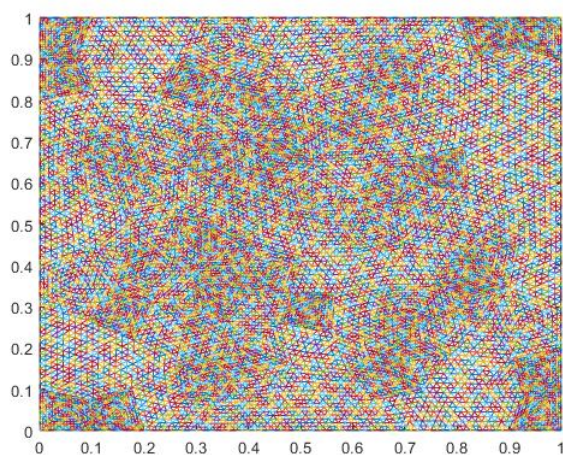
Obtaining a maximum error of 1.600364442024826e-04. With $h = 0.009760860118038$.

2.5   Refined mesh 4
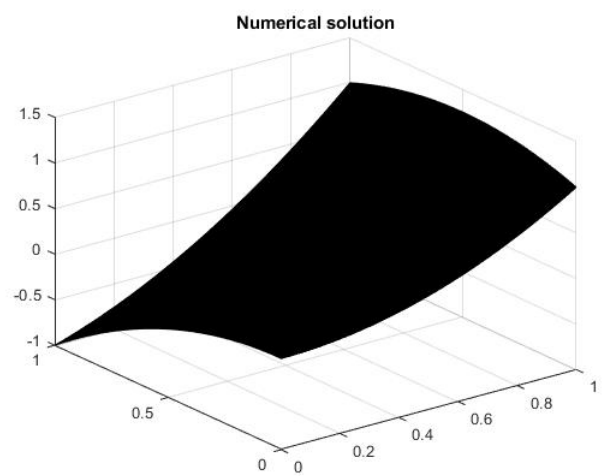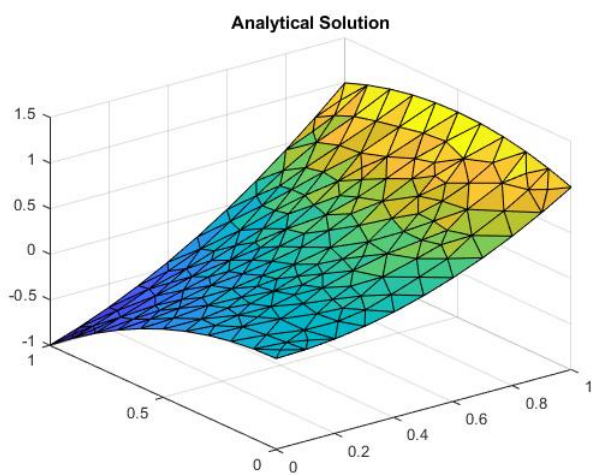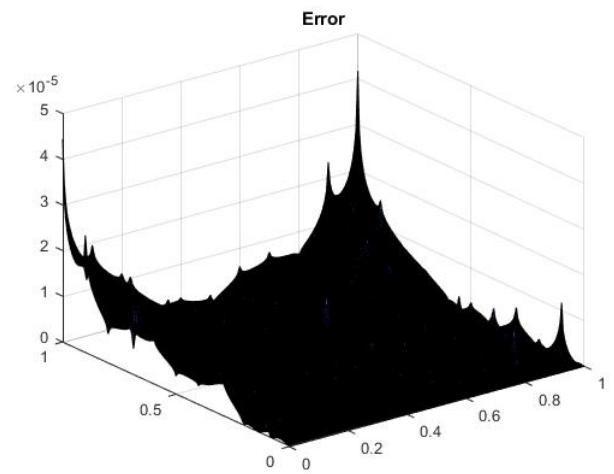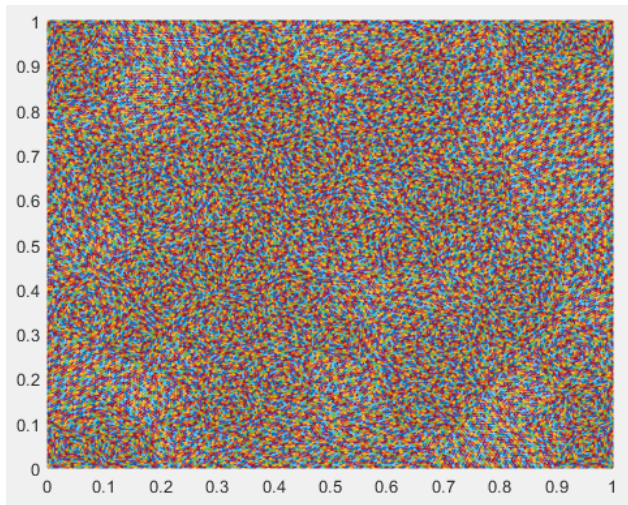
If we refine our mesh three times, we get the following results:

Obtaining a maximum error of 4.442343485666633e-05. With $h = 0.004880430059019$.

2.6 Theoretical Convergence

The results from the previous analysis can be founded in the next table:

h	Maximum error
0.078086880944303	0.006662135271408
0.039043440472152	0.001973332296642
0.019521720236076	5.678745361570314e-04
0.009760860118038	1.600364442024826e-04
0.004880430059019	4.442343485666633e-05

Table 1: Results summarize for different h

Now we are able to compute the logarithmic graph with to see if theoretical convergence holds, obtaining the following:
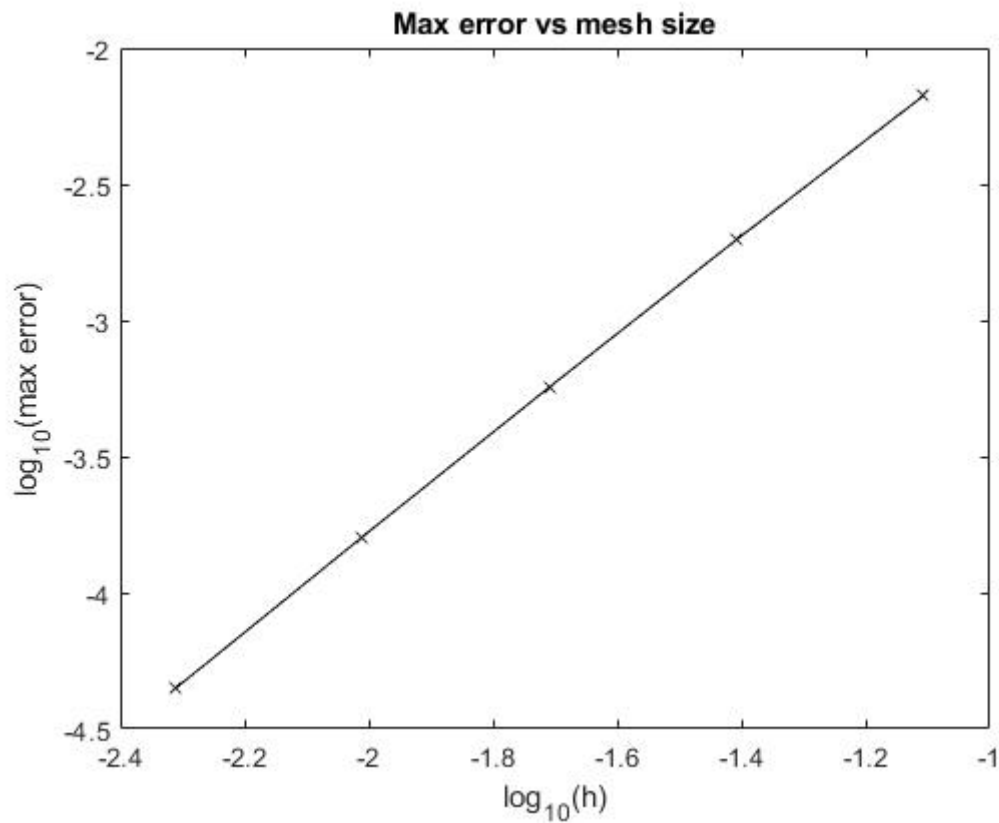
The mean slope we are suppose to obtain is 1.8, and for our results the mean slope was 1.807129972433525, meaning that the theoretical convergence holds.

3   Different times

If we solve the parabolic PDE for times 1,2,5,10 and 50 we get the following:

t	Maximum error
1	0.007388099174657
2	0.006949596783732
5	0.006649651169275
10	0.006662135271408
50	0.006662018152394

Table 2: Results summarize for different time

We can conclude that as time increases, the maximum error will remain almost the same, meaning that time doesn't significantly affect the solution.

4   Time dependence

It can be noticed in the analytical solution (8), that when $t_{end} = 50$ we can say that $e^{-3t} \simeq 0$, being (8) redefined as $u(x, y, t) = x^2 + xy - y^2$. Concluding that whenever we are dealing with a high value for t, f from (1) can be equal to 0 and the Dirichlet condition (6) can be written as $u(x, y = 0, t) = x^2$. Therefore it will not be a time variant partial differential equation, being (8) redefined as $-\Delta u = 0$. Notice that now we have an elliptic PDE.

Solving the PDE for 4 different meshes, we can now compare with the solutions for the parabolic PDE:

Elliptic max error	Parabolic max error
0.006662018152480	0.006662135271408
0.001973319082695	0.001973332296642
5.678757245873856e-04	5.678745361570314e-04
1.599212100564795e-04	1.600364442024826e-04

Table 3: Elliptic max error vs Parabolic max error

If we plot the logarithmic scale grapgh for the max error vs the size h of the mesh, we have:
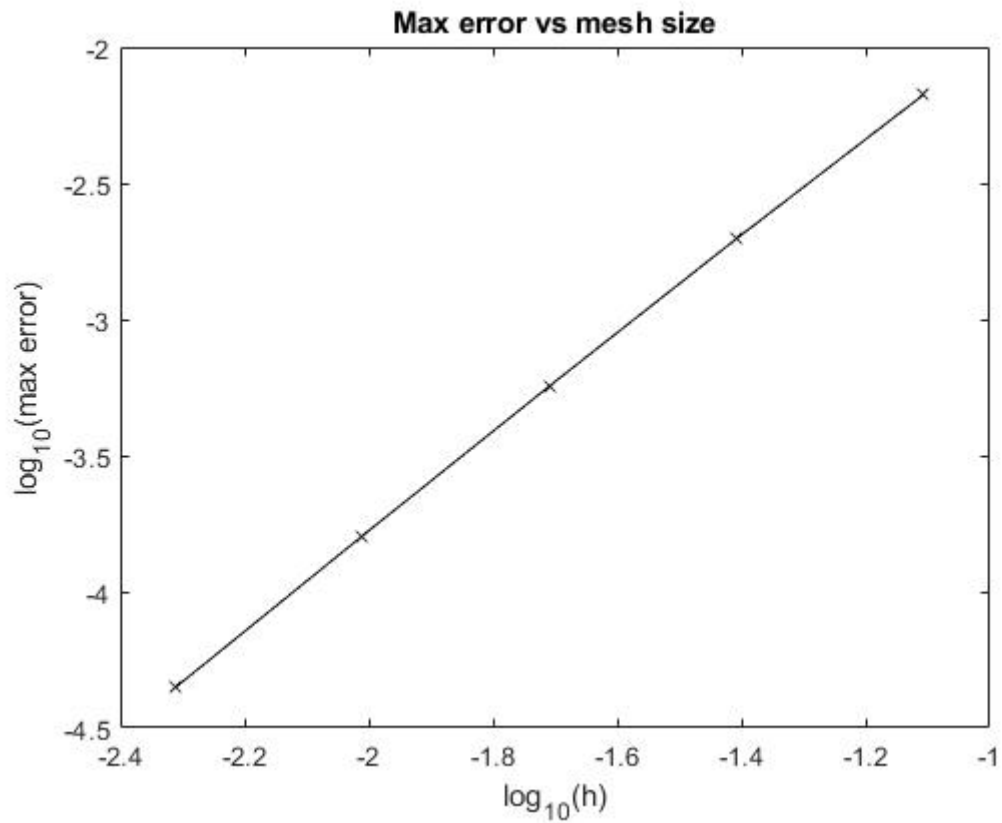
Obtaining a similar value for the mean slope of 1.807129972433525.