Assignment 2

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1 Problem:

The problem to solve is the parabolic PDE:

$$u_t - \triangle u = f \qquad in \quad \Omega =]0,1[^2$$

where the source term is given by:

$$f(x,y,t) = -3e^{-3t}.$$

with the initial condition:

$$u(x,y,0) = x^2 + xy - y^2 + 1$$

and the boundary conditions:

$$\begin{gathered} u_n(x=0,y,t) = -y \\ u_n(x=1,y,t) = 2+y \\ u(x,y=0,t) = x^2 + e^{-3t} \\ u_n(x,y=1,t) = x-2 \end{gathered}$$

2 Solution of the PDE:

The first step to solve the problem given using the *pdetool* of MATLAB was to define the geometry (a square of length 1). Once this was done we introduce the coefficients of our equation in the PDE toolbox. After this the boundary conditions (both Dirichlet and Neuman) were defined using the Boundary mode. Finally we introduce the Initial Condition in the Solve window and we left time to the default 0:10 as well as the Relative and Absolute tolerances. Once the problem was stated we created an initial mesh with the default parameters of the application. We saw that MATLAB created a mesh with 10 equally elements at the four boundaries. Finally we proceeded to solve the problem with this first coarse mesh and we exported the mesh and the solution to MATLAB Workspace. Then we repeated the same procedure with four meshes refinements and the solutions and meshes as well were ex-

Figure 1: Solution of the PDE using the initial mesh.Solution using the final mesh (below) after four refinements.

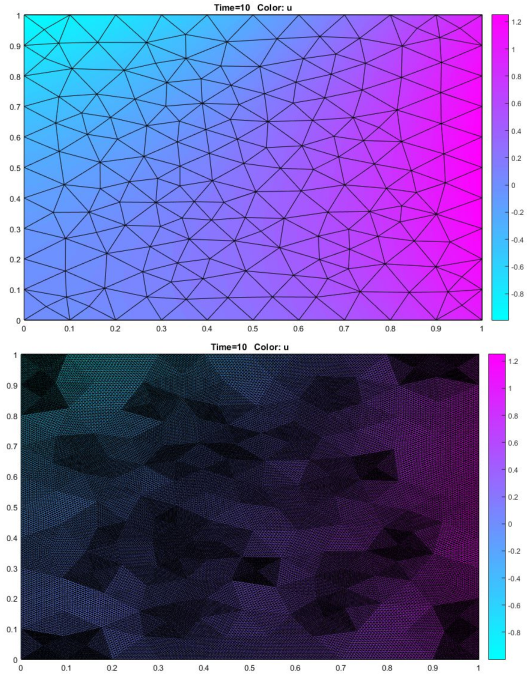
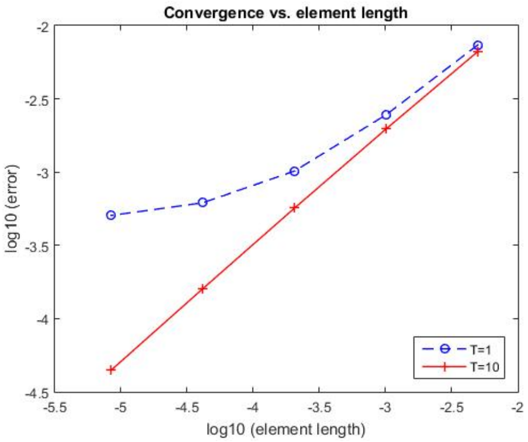


Figure 2: Convergence of the numerical solution with respect to the element length.



ported to the Workspace for the error and convergence analysis.

3 Element size effect:

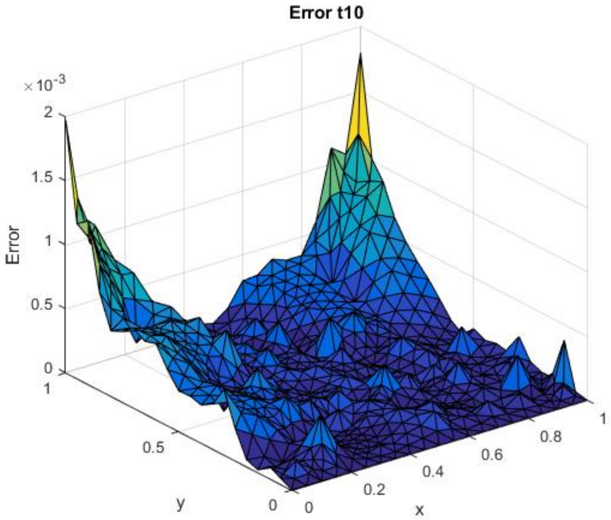
In order to analyse the convergence of the numerical solution computed with the *pdetool* to the analytical solution we modify the errorExample1 script in order to adapt it to our problem. What we did was to for each mesh discretization compute the maximum difference of the last time step [T=10] numerical solution respect the analytical solution. After that we plot the results in a loglog plot versus the element length measured at the lower boundary. Finally we compute the slopes of the convergence criteria to be compared with the theoretical ones. As it is seen in Figure 2 the convergence order is exponential because it is linear in a loglog plot, and this is the expected convergence because the mesh also is refined in a exponential way, with half the length of previous mesh.

Slopes of the convergence curves

Slope 1_1	1.583890175986223
Slope 1_{10}	1.755350730317373
Slope 2_1	1.277316808151475
Slope 2_{10}	1.796989790218398
Slope 3_1	0.720003372647428
Slope 3_{10}	1.827171742342429
Slope 4_1	0.286313049018653
Slope 4_{10}	1.849007626906428

If now we want to see how the error is distributed along the whole domain we must plot the error at each point, for instance for time T=10. This is what we show in Figure 3 for the first refinement of the mesh. The maximum error happens at the corners (0,1) and (1,1). This is because there we have prescribed the flux and the solution there is subjected to the errors of all the elements between them and the Dirichlet Boundary ($y = 0$) where the error is 0.

Figure 3: Error distribution of Mesh 2 at T=10.



4 Time effect:

In order to see how is the solution affected with an extension in the time end, we have repeated the calculations for the five meshes but for a final time one tenth the previous ones, T=1. Of course as this is a time dependent problem we will get a different solution. However we are interested in the precision and convergence order of the numerical solution for this time.

In theory what we should expect from this is that, as there are less calculations, the error will be higher and the convergence slower. This comparison can be seen in Figure 2 and it is clearly stated that for T=1 the convergence is slower than for T=10. Also in Table 3 it is shown that the slopes of T=1 are smaller than the ones of $T = 10$.

Now we want to solve the same problem but for time end T=50. One way of solving this is using the same PDE that we have computed for T=1 and T=10 and extending the calculations up to T=50. It maybe will take longer but it will work. However looking at the PDE, at T=50 the source term will tend to zero. This is because:

$$f(x, y, 50) = -3e^{-150} = -2.1525\times10^{-65} \approx 0.$$

Therefore, our problem can be simplified to:

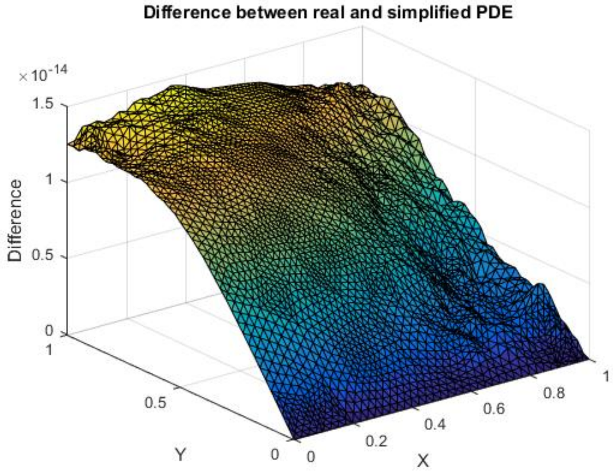
$$u_t - \triangle u = 0 \qquad in \quad \Omega =]0, 1[^2$$

Also the time dependent boundary condition can be expressed as:

$$u(x, y = 0, t) = x^2 + e^{-150} \approx x^2$$

This simplifications will reduce time of calculus and should approximate quite well our problem. We have done a comparison between the numerical solution of the real PDE and the simplified one for T=50. In Figure 4 we show that at T=50 there is very little difference in using the real PDE or the simplified one because it is of the order of $\times10^{-14}$. This evidences that the simplification we proposed could be used efficiently and that the result will be almost the same to the one of the extended PDE. Again the highest differences are at the opposed boundaries to where we have prescribed the values (Dirichlet boundary at y=0) because the error propagates throw the geometry.

Figure 4: Comparison between the numerical result of the real PDE and the simplified one for T=50 using Mesh 3.



5 Conclusions:

The PDE toolbox of MATLAB is an easy interface and pretty effective way of solving small partial differential equations problems. We have proved the effectiveness of this application by solving the same problem with different mesh sizes and it came clear that the convergence is proportional to the element size as expected. In addition we have checked that fewer time steps means fewer calculations so higher error at the end time and slower convergence. Finally we ran the code to solve the problem for T=50 and we show that for this problem and for long times, the solution will be almost stationary so we can dismiss the time dependence of the source term and of the boundary as well without effecting too much the solution.