The given function:

$$-\frac{\partial^2 u}{\partial x^2} = f$$

Boundary conditions:

u(0)=0 and u(1)=α

1. $\int_0^1 W_i \frac{\partial^2 u}{\partial x^2} dx = \int_0^1 W_i f dx$ where, W_i is the test function

Integrating by parts,

$$\int_0^1 \left\{\left[\frac{\partial^2 u}{\partial x^2}\right] \cdot W\right\} dx = W \frac{\partial u}{\partial x}\bigg|_0^1 + \int_0^1 \frac{\partial W}{\partial x}\frac{\partial u}{\partial x} dx$$

$$\int_0^1 \left\{\left[\frac{\partial^2 u}{\partial x^2}\right] \cdot W\right\} dx = \int_0^1 \frac{\partial W}{\partial x}\frac{\partial u}{\partial x} dx$$

$$\int_0^1 W \cdot f dx = \int_0^1 \frac{\partial W}{\partial x}\frac{\partial u}{\partial x} dx$$

Let,

$$U(x) = \sum_{i=0}^{1} N_i \cdot u_i$$

$W(x) = N_i$(x)

$$\int_0^1 \frac{\partial W}{\partial x}\frac{\partial u}{\partial x} dx = \int_0^1 W \cdot f dx + \text{w}.\frac{\partial u}{\partial x}\bigg|_0^1$$

Hence, the weak form of the equation is:

$$\boldsymbol{\int_0^1 \frac{\partial W}{\partial x}\frac{\partial u}{\partial x} dx = \int_0^1 W \cdot f dx + \text{w}.\frac{\partial u}{\partial x}\bigg|_0^1}$$

2. $\int_0^1 \frac{\partial W_i}{\partial x} \sum_{j=0}^{n} \frac{\partial N_j}{\partial x} a_j \, dx = \int_0^1 W_i \cdot f dx + [W_i . R]_1 + [W_i . R]_0$

$W_i = N_i$

$$\int_0^1 \frac{\partial N_i}{\partial x} \sum_{j=0}^{n} \frac{\partial N_j}{\partial x} a_j \, dx = \int_0^1 N_i \cdot f dx + [N_i . R]_1 + [N_i . R]_0$$

The global set equation

$$\int_0^1 \frac{\partial W_i}{\partial x} \left(\frac{\partial N_1}{\partial x} a_1 + \frac{\partial N_2}{\partial x} a_2 + \frac{\partial N_3}{\partial x} a_3 + \frac{\partial N_n}{\partial x} a_4 \right) dx = \int_0^1 W_1 \cdot f dx + [N_i . R]_1 + [N_i . R]_0$$

For i=1

$$\int_0^{1/3} \frac{\partial N_1^{(1)}}{\partial x} \left(\frac{\partial N_1^{(1)}}{\partial x} a_1 + \frac{\partial N_2^{(1)}}{\partial x} a_2 \right) dx = \int_0^{1/3} N_1^{(1)} \cdot f dx + R_0$$

For i=2

$$\int_0^{1/3} \frac{\partial N_2^{(1)}}{\partial x} \left(\frac{\partial N_1^{(1)}}{\partial x} a_1 + \frac{\partial N_2^{(1)}}{\partial x} a_2 \right) dx + \int_{1/3}^{2/3} \frac{\partial N_1^{(2)}}{\partial x} \left(\frac{\partial N_1^{(2)}}{\partial x} a_1 + \frac{\partial N_2^{(2)}}{\partial x} a_2 \right) dx = \int_{1/3}^{2/3} N_1^{(1)} \cdot f dx + \int_0^{1/3} N_1^{(2)} \cdot f dx$$

For i=3

$$\int_{1/3}^{2/3} \frac{\partial N_2^{(2)}}{\partial x} \left(\frac{\partial N_1^{(2)}}{\partial x} a_1 + \frac{\partial N_2^{(2)}}{\partial x} a_2 \right) dx + \int_{1/3}^{2/3} \frac{\partial N_1^{(3)}}{\partial x} \left(\frac{\partial N_1^{(3)}}{\partial x} a_1 + \frac{\partial N_2^{(3)}}{\partial x} a_2 \right) dx = \int_{1/3}^{2/3} N_1^{(1)} \cdot f dx + \int_0^{1/3} N_1^{(2)} \cdot f dx$$

For i=4

$$\int_{2/3}^{1} \frac{\partial N_2^{(3)}}{\partial x} \left(\frac{\partial N_1^{(3)}}{\partial x} a_1 + \frac{\partial N_2^{(3)}}{\partial x} a_2 \right) dx = \int_{2/3}^{1} N_2^{(3)} \cdot f dx + R_1$$

$$\begin{bmatrix} k_{11} & \cdots & k_{1n} \\ \vdots & \ddots & \vdots \\ k_{n1} & \cdots & k_{nn} \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$$

$$\boldsymbol{k} \cdot \boldsymbol{a} = \boldsymbol{f}$$

Shape functions can be generalized as:

$$N_1^{(e)}(x) = \frac{x_2^{(e)} - x}{l^{(e)}} \qquad ; \qquad N_2^{(e)}(x) = \frac{x - x_1^{(e)}}{l^{(e)}}$$

$$\frac{\partial {N_1}^{(1)}}{\partial x} = \frac{-1}{h^{(1)}} \qquad \frac{\partial {N_1}^{(2)}}{\partial x} = \frac{-1}{h^{(2)}} \qquad \frac{\partial {N_1}^{(3)}}{\partial x} = \frac{-1}{h^{(3)}}$$

$$\frac{\partial {N_2}^{(1)}}{\partial x} = \frac{1}{h^{(1)}} \qquad \frac{\partial {N_2}^{(2)}}{\partial x} = \frac{1}{h^{(2)}} \qquad \frac{\partial {N_2}^{(3)}}{\partial x} = \frac{1}{h^{(3)}}$$

$$k_{11} = \int_0^{1/3} \frac{\partial {N_1}^{(1)}}{\partial x}\frac{\partial {N_1}^{(1)}}{\partial x} dx = \frac{1}{h^{2(1)}} \cdot \frac{1}{3} = \frac{1}{3h^{2(1)}} = 3$$

$$k_{12} = \int_0^{1/3} \frac{\partial {N_1}^{(1)}}{\partial x}\frac{\partial {N_2}^{(1)}}{\partial x} dx = \frac{1}{3h^{3(2)}} = -3$$

$$k_{22} = \int_0^{1/3} \frac{\partial {N_2}^{(1)}}{\partial x}\frac{\partial {N_2}^{(1)}}{\partial x} dx = \frac{1}{3h^{2(2)}} = \frac{1}{3({}^{1}/_{3})^2} = 3$$

$k_{11} \quad = k_{22}$

$k_{12} \quad = k_{21}$

3. **Given,**

$f(x) = \ sin(x)\ ; \alpha = 3$

$${f_1}^{(1)} \qquad = \int_0^{1/3} {N_1}^{(1)} \cdot f(x)\, dx$$

$$= \int_0^{1/3} \frac{{x_2}^{(1)} - x}{h^{(1)}} \cdot sin(x)\, dx = \int_0^{1/3} \frac{{x_2}^{(1)}}{h^{(1)}} \cdot sin(x)\, dx + \int_0^{1/3} \frac{-x}{h^{(1)}} \cdot sin(x)\, dx$$

$$= -\frac{{x_2}^{(1)}}{h^{(1)}} \cdot cos(x)\Big|_0^{1/3} - \frac{1}{h^{(1)}} \cdot (\sin(x) - x \cdot cos(x)\Big|_0^{1/3}$$

$= -1 - 3 \cdot \sin({}^{1}/_{3})$ = 0.0184

$${f_2}^{(1)} \qquad = \int_0^{1/3} {N_2}^{(1)} \cdot f(x)\, dx = \int_0^{1/3} \frac{x - {x_1}^{(1)}}{h^{(1)}} \cdot sin(x)\, dx$$

$= 3\left(\sin({}^{1}/_{3}) - {}^{1}/_{3} \cdot cos({}^{1}/_{3})\right) = 3 \cdot \sin({}^{1}/_{3}) - cos({}^{1}/_{3})$= 0.0366

$$f_1^{(2)} = \int_{\frac{1}{3}}^{2/3} N_1^{(2)} \cdot f(x)\,dx = \int_{\frac{1}{3}}^{2/3} \frac{x_3 - x}{h} \cdot sin(x)\,dx$$

$$= -\frac{x_3}{h} \cdot cos(x)\Big|_{\frac{1}{3}}^{2/3} - \frac{1}{h} \cdot (\sin(x) - x \cdot cos(x)\Big|_{\frac{1}{3}}^{\frac{2}{3}}$$

$$= -2\left[cos\left(\frac{2}{3}\right) - cos\left(\frac{1}{3}\right)\right] - 3\left[sin\left(\frac{2}{3}\right) - \frac{2}{3}cos\left(\frac{1}{3}\right) - sin\left(\frac{1}{3}\right) + \frac{1}{3}cos\left(\frac{1}{3}\right)\right]$$

$$= 2cos\left(\frac{1}{3}\right) - 3sin\left(\frac{2}{3}\right) + 3sin\left(\frac{1}{3}\right) - \cos\left(\frac{1}{3}\right) = 0.0714$$

$$f_2^{(2)} = \int_{\frac{1}{3}}^{2/3} N_1^{(2)} \cdot f(x)\,dx = \int_{\frac{1}{3}}^{2/3} \frac{-x_2}{h} \cdot sin(x)\,dx + \int_{\frac{1}{3}}^{2/3} \frac{x}{h} \cdot sin(x)\,dx$$

$$= -\frac{x_2}{h} \cdot cos(x)\Big|_{\frac{1}{3}}^{2/3} + \frac{1}{h} \cdot (\sin(x) - x \cdot cos(x)\Big|_{\frac{1}{3}}^{\frac{2}{3}}$$

$$= -\cos\left(\frac{2}{3}\right) - \cos\left(\frac{1}{3}\right) + 3sin\left(\frac{2}{3}\right) - \left(\frac{2}{3}\right)cos\left(\frac{2}{3}\right) - sin\left(\frac{1}{3}\right) + \frac{1}{3}cos\left(\frac{1}{3}\right)$$

$$= 2cos\left(\frac{1}{3}\right) - cos\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)cos\left(\frac{2}{3}\right) - 3sin\left(\frac{1}{3}\right) - 2\cos\left(\frac{2}{3}\right)$$

$$= cos\left(\frac{2}{3}\right) + 3sin\left(\frac{2}{3}\right) - 3sin\left(\frac{1}{3}\right) = 0.0877$$

$f_1^{(3)} = \int_{\frac{2}{3}}^{1} \frac{x_4}{h} \cdot sin(x)\,dx + \int_{\frac{2}{3}}^{1} \frac{-x}{h} \cdot sin(x)\,dx = -\frac{x_4}{h} \cdot cos(x)\Big|_{\frac{2}{3}}^{1} - \frac{1}{h} \cdot (\sin(x) - x \cdot cos(x)\Big|_{\frac{2}{3}}^{1} = -3\cos(1) + 3\cos(1) + 3\cos\left(\frac{2}{3}\right) - 3\left[sin(1) - cos(1) - sin\left(\frac{2}{3}\right) + \frac{2}{3}cos\left(\frac{2}{3}\right)\right] = \cos\left(\frac{2}{3}\right) - 3\sin(1) + 3\sin\left(\frac{2}{3}\right) = 0.1163$

$f_2^{(3)} = \int_{\frac{2}{3}}^{1} \frac{-x_3}{h} \cdot sin(x)\,dx + \int_{\frac{2}{3}}^{1} \frac{x}{h} \cdot sin(x)\,dx = \frac{x_3}{h} \cdot cos(x)\Big|_{\frac{2}{3}}^{1} + \frac{1}{h} \cdot (\sin(x) - x \cdot cos(x)\Big|_{\frac{2}{3}}^{1} = 2\cos(1) - \cos\left(\frac{2}{3}\right) + 3[sin(1) - cos(1)] - \sin\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)\cos\left(\frac{2}{3}\right) = -\cos(1) + 3\sin(1) - 3sin\left(\frac{2}{3}\right) = 0.1291$

The matrix reads

R_0= c, R_1= d ,u =a_2 , v= a_3

$$3\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ \mathrm{u} \\ v \\ 3 \end{bmatrix} = \begin{bmatrix} 0.0184 + c \\ 0.108 \\ 0.2042 \\ 0.1291 + d \end{bmatrix}$$

$3(2a_2 - a_3) = 0.108$ $(equ\ 1)$

$3(-a + 2a_2) - 9 = 0.2043\ (equ\ 2)$

$equ\ 1 + \ equ\ 2 * \ equ\ 2$

$9\ a_3 = 18.5164$

$a_3 = 2.0573$

$6a_2 = 0.108 + 3\ a_3$

$a_2 = 1.0466$

exact solution

u(x)= sin x + 3 (sin 1) x

for u(0) = 1.0467

for u(1/3) = 1.0467

for u(2/3) = 2.0573

for u(1) = 3

hence,

FEM Solution = Exact Solution