FEM

Homework - 2

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$t = 1m$, Young's modulus, $E = 10\,GPa$, Poisson Ratio, $\nu = 0$

vertical displacement $\delta = 10^{-2} = 0.01\,m$, Body force $= \rho g = 10^{3}\,N/m^2$

Strong form

$$b_x + \frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0$$

$$b_y + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} = 0$$

Boundary Conditions

From the fig. given, it can be seen

$u_1 = u_2 = u_3 = v_1 = v_2 = v_3 = 0$ (Fixed nodes 1, 2, 3)

$u_5 = 0$ (Symmetry condition)

$\delta = 10^{-2} = 0.01\,m = v_6$ (given)

Nodal Coordinates & Connectivity matrix

Nodes	x	y
1	-3	0
2	1.5	0
3	0	0
4	-1.5	1.5
5	0	1.5
6	0	3

~~Nod~~ Nodal cooridinates (X)

	Nodes		
Element	1	2	3
1	2	4	1
2	4	2	5
3	3	5	2
4	5	6	4

T - Matrix

Mesh Description – In order to make the discritization easy local numbering is made such that in every element, the node in the right angle vertex has a local no. equal to 1

To find the discritization of displacement field, we have

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$
$$v = N_1 v_1 + N_2 v_2 + N_3 v_3$$

The 3 nodes of triangular mesh defines linear displacement field which is written as

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y$$
$$v = \alpha_4 + \alpha_5 x + \alpha_6 y$$

After deriving the shape f^n for u alone

$$u_1 = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1$$
$$u_2 = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2$$
$$u_3 = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3$$

$$N_i = \frac{1}{2A^{(e)}} (a_i + b_i x + c_i y)$$

where $a_i = x_j y_k - x_k y_j$; $b_i = y_j - y_k$

$c_i = x_k - x_j$

Stiffness matrix $K^e = \iint_{A^e} B^T D B \, t \, dA$

Equivalent Nodal force vector

$$f^e = f_\epsilon^e + f_\sigma^e + f_b^e + f_t^e$$

$$f_\epsilon^e = \iint_{A^e} B^T D \epsilon^e t \, dA$$

$$f_\sigma^e = \iint_{A^e} B^T D \sigma^e t \, dA$$

$$f_b^e = \iint_{A^e} N^T b \, t \, dA$$

$$f_t^e = \iint_{A^e} N^T t \, t \, dA$$

$$K_{ij} = \left(\frac{t}{4A}\right)^e \begin{bmatrix} b_i b_j d_{11} + c_i c_j d_{33} & b_i c_j d_{12} + b_j c_i d_{33} \\ c_i b_j d_{21} + b_i c_j d_{33} & b_i b_j d_{33} + c_i c_j d_{22} \end{bmatrix}$$

As we are dealing with plane stress problem, we have

$$\sigma = D\epsilon$$

D is the constitutive matrix $D = \begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{21} & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}$

$$d_{11} = d_{22} = \frac{E}{1-\nu^2} = \frac{10\ GPa}{1-0.2^2} = 10.417\ GPa$$

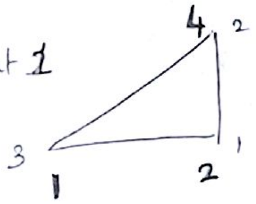
$$d_{12} = d_{21} = \nu d_{11} = 0.2 \times 10.417 = 2.083\ GPa$$

$$d_{33} = \frac{E}{2(1+\nu)} = \frac{10}{2(1+0.2)} = 4.167\ GPa$$

1, 2, 3 → Local numbering

1, 2, 4 → Global numbering

Element 1



$$(x_1, y_1)' = (-1.5, 0)$$
$$(x_2, y_2)' = (-1.5, 1.5)$$
$$(x_3, y_3)' = \cancel{(3, 0)}\ (-3, 0)$$

$$b_i = y_j - y_k \quad ; \quad c_i = x_k - x_j$$

$$b_1 = y_2 - y_3 = 1.5 \quad ; \quad c_1 = x_3 - x_2 = -1.5$$
$$b_2 = y_3 - y_1 = 0 \quad ; \quad c_2 = x_1 - x_3 = 1.5$$
$$b_3 = y_1 - y_2 = -1.5 \quad ; \quad c_3 = x_2 - x_1 = 0$$

$$\therefore K_{ij} = \left(\frac{t}{4A}\right)^e \begin{bmatrix} b_i b_j d_{11} + c_i c_j d_{33} & b_i c_j d_{12} + b_j c_i d_{33} \\ c_i b_j d_{21} + b_i c_j d_{33} & b_i b_j d_{33} + c_i c_j d_{22} \end{bmatrix}$$

$$k_{11}^{①} = \frac{2}{9} \begin{bmatrix} 2.25 \times 10.417 + 2.25 \times 4.167 & -2.25 \times 2.083 - 2.25 \times 4.167 \\ -2.25 \times 2.083 + (-2.25) \times 4.167 & 2.25 \times 10.417 + 2.25 \times 4.167 \end{bmatrix}$$

$$= \begin{bmatrix} 7.292 & -3.125 \\ -3.125 & 7.292 \end{bmatrix}$$

Similarly,

$$k_{12}^{(1)} = \frac{2\times 2.25}{9}\begin{bmatrix} -4.167 & 2.083 \\ 4.167 & -10.417 \end{bmatrix} = \begin{bmatrix} -2.083 & 1.042 \\ 2.083 & -5.2085 \end{bmatrix}$$

$$= K_{21}^{e}$$

$$K_{13}^{(1)} = \frac{2\times 2.25}{9}\begin{bmatrix} -10.417 & 4.167 \\ 2.083 & -4.167 \end{bmatrix} = \begin{bmatrix} -5.2085 & 2.083 \\ 1.042 & -2.083 \end{bmatrix} = K_{31}^{e}$$

$$K_{22}^{(1)} = \frac{2\times 2.25}{9}\begin{bmatrix} 4.167 & 0 \\ 0 & 10.417 \end{bmatrix} = \begin{bmatrix} 2.083 & 0 \\ 0 & 5.2085 \end{bmatrix}$$

$$K_{23}^{(1)} = \frac{2\times 2.25}{9}\begin{bmatrix} 0 & -4.167 \\ -2.083 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2.083 \\ -1.042 & 0 \end{bmatrix} = K_{32}^{e}$$

$$K_{33}^{(1)} = \frac{2\times 2.25}{9}\begin{bmatrix} 10.417 & 0 \\ -2.083 & 4.167 \end{bmatrix} = \begin{bmatrix} 5.2085 & 0 \\ 0 & 2.083 \end{bmatrix}$$

$$K_{11}^{(3)} = k_{11}^{(4)} = K_{11}^{(1)} \; ; \quad K_{12}^{(3)} = k_{12}^{(4)} = K_{12}^{(1)}$$

$$K_{13}^{(3)} = K_{13}^{(4)} = K_{13}^{(1)}$$

$$K_{21}^{(3)} = k_{21}^{(4)} = K_{21}^{(1)} \; ; \quad K_{22}^{(3)} = k_{22}^{(4)} = K_{22}^{(1)}$$

$$K_{23}^{(3)} = K_{23}^{(4)} = K_{23}^{(1)}$$

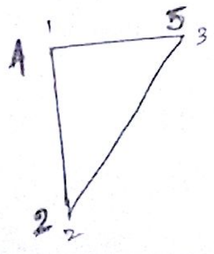
$$K_{31}^{(3)} = K_{31}^{(4)} = K_{31}^{(1)} \; ; \quad K_{32}^{(3)} = k_{32}^{(4)} = K_{32}^{(1)}$$

$$K_{33}^{(3)} = K_{33}^{(4)} = K_{33}^{(1)}$$

Therefore element 1, 3 & 4 have same stiffness matrix

For element 2, to compute stiffness matrix;

From nodal coordinates and considering local numbering, we have



Element 2

$(x_1, y_1)^{(2)} = (-1.5, 1.5)$

$(x_2, y_2)^{(2)} = (-1.5, 0)$

$(x_3, y_3)^{(2)} = (0, 1.5)$

$b_i = y_j - y_k \qquad c_i = x_k - x_j$

$b_1 = y_2 - y_3 = -1.5 \quad ; \quad c_1 = x_3 - x_2 = 1.5$

$b_2 = y_3 - y_1 = 0 \quad ; \quad c_2 = x_1 - x_3 = -1.5$

$b_3 = y_1 - y_2 = 1.5 \quad ; \quad c_3 = x_2 - x_1 = 0$

Since the values are same as that of element 1, but opposite in sign. Hence we have the same stiffness matrix for the element 2 as well,

Therefore $k^1 = k^2 = k^3 = k^4$

Assembly of Stiffness matrix

$$K^1 = \begin{bmatrix} K_{11}^1 & K_{12}^1 & K_{13}^1 \\ K_{21}^1 & K_{22}^1 & K_{23}^1 \\ K_{31}^1 & K_{32}^1 & K_{33}^1 \end{bmatrix} = K^2 = \begin{bmatrix} K_{11}^2 & K_{12}^2 & K_{13}^2 \\ K_{21}^2 & K_{22}^2 & K_{23}^2 \\ K_{31}^2 & K_{32}^2 & K_{33}^2 \end{bmatrix} =$$

$$K^3 = \begin{bmatrix} K_{11}^3 & K_{12}^3 & K_{13}^3 \\ K_{21}^3 & K_{22}^3 & K_{23}^3 \\ K_{31}^3 & K_{32}^3 & K_{33}^3 \end{bmatrix} = K^4 = \begin{bmatrix} K_{11}^4 & K_{12}^4 & K_{13}^4 \\ K_{21}^4 & K_{22}^4 & K_{23}^4 \\ K_{31}^4 & K_{32}^4 & K_{33}^4 \end{bmatrix}$$

The assembly of matrix is :

$$K = \begin{bmatrix} K_{33}^{(1)} & K_{13}^{(1)T} & 0 & K_{23} & 0 & 0 \\ & K_{11}^{(1)} + K_{22}^{(1)} + K_{33}^{(1)} & K_{13}^{(3)T} & K_{12}^{(1)} + K_{12}^{(2)T} & K_{23}^{(2)} + K_{23}^{(3)T} & 0 \\ & & K_{11}^{(3)} & 0 & K_{12}^{(3)} & 0 \\ & & & K_{22}^{(1)} + K_{11}^{(2)} + K_{33}^{(4)} & K_{13}^{(2)} + K_{13}^{(4)T} & K_{23}^{(4)T} \\ & & & & K_{11}^{(4)} + K_{22}^{(3)} + K_{33}^{(2)} & K_{12}^{(4)} \\ & & & & & K_{22}^{(4)} \end{bmatrix}$$

We know that

$$a_i^e = \begin{bmatrix} u_i \\ v_i \end{bmatrix} \Rightarrow a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}$$

for the given problem, the displacement matrix 'a' is given by

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}$$

The nodal force vector is :

$$f = \begin{bmatrix} f_3^{(1)} + r_1 \\ f_1^{(1)} + f_2^{(2)} + f_3^{(3)} + r_2 \\ f_1^{(3)} + r_3 \\ f_2^{(1)} + f_1^{(2)} + f_3^{(4)} \\ f_3^{(2)} + f_2^{(3)} + f_1^{(4)} \\ f_2^{(4)} \end{bmatrix}$$

Therefore $Ka = f$

The given problem has 12 degrees of freedom, where in 9 degrees are constrained.

To Compute Nodal displacements

From B.C. we have $u_1 = u_2 = u_3 = v_1 = v_2 = v_3 = 0$

hence $a_1 = a_2 = a_3 = 0$

Hence we need to consider only 4, 5 & 6 rows only from global stiffness matrix

and also $u_5 = u_6 = 0$

Therefore,

$$\begin{bmatrix} K_{22}^{(1)} + K_{11}^{(2)} + K_{3}^{(4)} & K_{13}^{(2)} + K_{13}^{(4)T} & K_{23}^{(4)T} \\ & K_{11}^{(4)} + K_{22}^{(3)} + K_{33}^{(2)} & K_{12}^{(4)} \\ & & K_{22}^{(4)} \end{bmatrix} \begin{bmatrix} u_4 \\ v_4 \\ u_5 = 0 \\ v_5 \\ u_6 = 0 \\ v_6 = \delta = 10^{-2} \end{bmatrix} = \begin{bmatrix} f_2^{(1)} + f_1^{(2)} + f_3^{(4)} \\ f_3^{(2)} + f_2^{(3)} + f_1^{(4)} \\ f_2^{(4)} \end{bmatrix} \quad -(1)$$

Now substituting value of k_{ij}^{e} for the above matrix, we get

$$K = \begin{bmatrix} -14.5835 & -3.125 & -10.417 & 3.125 & 0 & 1.042 \\ -3.125 & 14.5835 & 4.166 & -4.166 & -1.042 & 0 \\ -10.417 & 4.166 & 14.5835 & -3.125 & -2.083 & 1.042 \\ 3.125 & -4.166 & -3.125 & 14.5835 & 2.083 & -5.2085 \\ 0 & -1.042 & -2.083 & 2.083 & 2.083 & 0 \\ 1.042 & 0 & 1.042 & -5.2085 & 0 & 5.2085 \end{bmatrix} \text{GN/m}$$

We have from data that the whole domain deforms because of self weight with gravity acting in the direction y-axis. Therefore only body forces are significant and not surface loads.

Body forces for the equivalent nodal force is given by

$$f_{b_i} = \left(\frac{At}{3}\right)^e \begin{bmatrix} b_x \\ b_y \end{bmatrix}$$

$$b_x = 0, \; b_y = -\rho g = -10^{-3}$$

$$f_{b_i} = \left(\frac{2.25}{6}\right) \begin{bmatrix} 0 \\ -10^{3} \end{bmatrix} = \begin{bmatrix} 0 \\ -375 \end{bmatrix} N$$

Substituting $f_{b_i}^e$ & K in eq. (1) we get by simplifying.

$$\begin{bmatrix} 14.5835 & -3.125 & 3.125 & 1.042 \\ -3.125 & 14.5835 & -4.166 & 0 \\ 3.125 & -4.166 & 14.5835 & -5.2085 \\ 1.042 & 0 & -5.2085 & 5.2085 \end{bmatrix} \times 10^{9} \begin{bmatrix} u_4 \\ v_4 \\ v_5 \\ -10^{-2} \end{bmatrix} = \begin{bmatrix} 0 \\ -1125 \\ -1125 \\ -375 \end{bmatrix}$$

Solving the above system of linear equations, we get the nodal displacement values as

$$\boxed{\begin{aligned} u_4 &= -1.29 \times 10^{-4} \\ v_4 &= -1.13 \times 10^{-3} \\ v_5 &= -3.87 \times 10^{-3} \end{aligned}}$$