Homework 2

Plane Elasticity

Finite Elements Method

Ana Salas Ordóñez
DNI 77196118E

22 December 2015

1. Describe the strong form of the problem in the reduced domain (left half). Indicate accurately the Boundary Conditions in every edge.

Equilibrium equation

$$\text{div } \sigma + b = 0; \quad \begin{pmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{pmatrix} + \begin{pmatrix} b_x \\ b_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Constitutive equation

$$\sigma = D\varepsilon; \quad \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{pmatrix} = \begin{pmatrix} \frac{E}{1-\nu^2} & \nu\frac{E}{1-\nu^2} & 0 \\ \nu\frac{E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & \frac{E}{2(1-\nu)} \end{pmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix}$$

Compatibility equation

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix}$$

Strong form

$$\begin{pmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} \frac{E}{1-\nu^2} & \nu\frac{E}{1-\nu^2} & 0 \\ \nu\frac{E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & \frac{E}{2(1-\nu)} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} + \begin{pmatrix} b_x \\ b_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Boundary conditions

$$u_1 = v_1 = u_2 = v_2 = u_3 = v_3 = u_5 = u_6 = 0$$

$$v_6 = -\delta$$

2. Describe the mesh shown in figure 2 by giving the arrays of nodal coordinates and the connectiviy matrix T.

Following the element numbering shown in figure 2, we can write the connectivity matrix as follows, being the columns the number of the nodes of each element:

$$\text{T}=\begin{bmatrix} 2 & 4 & 1 \\ 4 & 2 & 5 \\ 3 & 5 & 2 \\ 5 & 6 & 4 \end{bmatrix}$$

We describe below the arrays of nodal coordinates X and Y:

$$\text{X}=\begin{bmatrix} 1.5 & 1.5 & 0 \\ 1.5 & 1.5 & 3 \\ 3 & 3 & 1.5 \\ 3 & 3 & 1.5 \end{bmatrix} \qquad \text{Y}=\begin{bmatrix} 0 & 1.5 & 0 \\ 1.5 & 0 & 1.5 \\ 0 & 1.5 & 0 \\ 1.5 & 3 & 1.5 \end{bmatrix}$$

3. Set up the linear system of equations corresponding to the discretization in figure 2. How many degrees of freedom has the system to be solved?.

The global system of equations has the following form:

$$\underline{\mathbf{K}}\underline{\mathbf{a}}=\underline{\mathbf{f}}$$

$$\mathbf{K}=\begin{bmatrix} \mathbf{K}_{33}^{(1)} & \mathbf{K}_{13}^{(1)} & \mathbf{0} & \mathbf{K}_{23}^{(1)} & \mathbf{0} & \mathbf{0} \\ & \mathbf{K}_{11}^{(1)}+\mathbf{K}_{22}^{(2)}+\mathbf{K}_{33}^{(3)} & \mathbf{K}_{31}^{(3)} & \mathbf{K}_{12}^{(1)}+\mathbf{K}_{21}^{(2)} & \mathbf{K}_{23}^{(2)}+\mathbf{K}_{32}^{(3)} & \mathbf{0} \\ & & \mathbf{K}_{11}^{(3)} & \mathbf{0} & \mathbf{K}_{12}^{(3)} & \mathbf{0} \\ & & & \mathbf{K}_{22}^{(1)}+\mathbf{K}_{11}^{(2)}+\mathbf{K}_{33}^{(4)} & \mathbf{K}_{13}^{(2)}+\mathbf{K}_{31}^{(4)} & \mathbf{K}_{32}^{(4)} \\ & \mathit{Symm.} & & & \mathbf{K}_{33}^{(2)}+\mathbf{K}_{22}^{(3)}+\mathbf{K}_{11}^{(4)} & \mathbf{K}_{12}^{(4)} \\ & & & & & \mathbf{K}_{22}^{(4)} \end{bmatrix}$$

$$\text{a}^T=[u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3 \quad u_4 \quad v_4 \quad u_5 \quad v_5 \quad u_6 \quad v_6]$$

$$\mathrm{f}=\begin{Bmatrix} f_{3x}^{(1)}+r_{1x} \\ f_{3y}^{(1)}+r_{1y} \\ f_{1x}^{(1)}+f_{2x}^{(2)}+f_{3x}^{(3)}+r_{2x} \\ f_{1y}^{(1)}+f_{2y}^{(2)}+f_{3y}^{(3)}+r_{2y} \\ f_{1x}^{(3)}+r_{3x} \\ f_{1y}^{(3)}+r_{3y} \\ f_{2x}^{(1)}+f_{1x}^{(2)}+f_{3x}^{(4)} \\ f_{2y}^{(1)}+f_{1y}^{(2)}+f_{3y}^{(4)} \\ f_{3x}^{(2)}+f_{2x}^{(3)}+f_{1x}^{(4)}+r_{5x} \\ f_{3y}^{(2)}+f_{2y}^{(3)}+f_{1y}^{(4)} \\ f_{2x}^{(4)}+r_{6x} \\ f_{2y}^{(4)}+r_{6y} \end{Bmatrix}$$

$$u_1=v_1=u_2=v_2=u_3=v_3=u_5=u_6=0$$

$$v_6=-\delta$$

The system has two degrees of freedom, u and v.

4. Compute the FE approximation u^h. Use E=10 GPa, $\nu=0.2, \delta=10^{-2}m$ and $\rho g=10^3 N/m^2$.

The analysis domain is discretized into a mesh of four 3-noded triangular elements as shown

$$\mathrm{K}_{ij}^{(e)}=(\tfrac{t}{4A})^{(e)}\begin{bmatrix} b_ib_jd_{11}+c_ic_jd_{33} & b_ic_jd_{12}+b_jc_id_{33} \\ c_ib_jd_{21}+b_ic_jd_{33} & b_ib_jd_{33}+c_ic_jd_{22} \end{bmatrix}$$

with:

$$\mathrm{b}_i=y_j^{(e)}-y_k^{(e)};\quad b=\begin{bmatrix} b_1^{(1)} & b_2^{(1)} & b_3^{(1)} \\ b_1^{(2)} & b_2^{(2)} & b_3^{(2)} \\ b_1^{(3)} & b_2^{(3)} & b_3^{(3)} \\ b_1^{(4)} & b_2^{(4)} & b_3^{(4)} \end{bmatrix}=\begin{bmatrix} 1.5 & 0 & -1.5 \\ -1.5 & 0 & 1.5 \\ 1.5 & 0 & -1.5 \\ 1.5 & 0 & -1.5 \end{bmatrix}$$

$$\mathrm{c}_i=x_k^{(e)}-x_j^{(e)};\quad c=\begin{bmatrix} c_1^{(1)} & c_2^{(1)} & c_3^{(1)} \\ c_1^{(2)} & c_2^{(2)} & c_3^{(2)} \\ c_1^{(3)} & c_2^{(3)} & c_3^{(3)} \\ c_1^{(4)} & c_2^{(4)} & c_3^{(4)} \end{bmatrix}=\begin{bmatrix} -1.5 & 1.5 & 0 \\ 1.5 & -1.5 & 0 \\ -1.5 & 1.5 & 0 \\ -1.5 & 1.5 & 0 \end{bmatrix}$$

$$\mathrm{d}=\begin{bmatrix} \frac{E}{1-\nu^2} & \nu\frac{E}{1-\nu^2} & 0 \\ \nu\frac{E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & \frac{E}{2(1-\nu)}=G \end{bmatrix}=\begin{bmatrix} 1.0417\times10^{10} & 2.0833\times10^{9} & 0 \\ 2.0833\times10^{9} & 1.0417\times10^{10} & 0 \\ 0 & 0 & 4.1667\times10^{9} \end{bmatrix}$$

Substituting the previous values of b, c and d for each element, we obtain the local stifness and force matrix of each element:

$$K^{(1)} = K^{(2)} = K^{(3)} = K^{(4)} = \begin{bmatrix} K_{11}^{(e)} & K_{12}^{(e)} & K_{13}^{(e)} \\ K_{21}^{(e)} & K_{22}^{(e)} & K_{23}^{(e)} \\ K_{31}^{(e)} & K_{32}^{(e)} & K_{33}^{(e)} \end{bmatrix}$$

$$\mathrm{K}^{(e)} = 10^9 \begin{bmatrix} 7.2917 & -3.1250 & -2.0833 & 1.0417 & -5.2083 & 2.0833 \\ -3.1250 & 7.2917 & 2.0833 & -5.2083 & 1.0417 & -2.0833 \\ -2.0833 & 2.0833 & 2.0833 & 0 & 0 & -2.0833 \\ 1.0417 & -5.2083 & 0 & 5.2083 & -1.0417 & 0 \\ -5.2083 & 1.0417 & 0 & -1.0417 & 5.2083 & 0 \\ 2.0833 & -2.0833 & -2.0833 & 0 & 0 & 2.0833 \end{bmatrix}$$

$$f_{bi} = \frac{At)}{3}^{(e)} \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} \; ; \; \mathrm{f}_b = \frac{(At)}{3} \begin{Bmatrix} f_1^{(1)} & f_2^{(1)} & f_3^{(1)} \\ f_1^{(2)} & f_2^{(2)} & f_3^{(2)} \\ f_1^{(3)} & f_2^{(3)} & f_3^{(3)} \\ f_1^{(4)} & f_2^{(4)} & f_3^{(4)} \end{Bmatrix} = \begin{Bmatrix} 0 & 0 & 0 \\ -375 & -375 & -375 \\ 0 & 0 & 0 \\ -375 & -375 & -375 \\ 0 & 0 & 0 \\ -375 & -375 & -375 \\ 0 & 0 & 0 \\ -375 & -375 & -375 \end{Bmatrix}$$

Global stifness matrix:

$$K = 10^{10} \begin{bmatrix} 0.5208 & 0 & -0.5208 & 0.1042 & 0 & 0 & 0 & -1.042 & 0 & 0 & 0 & 0 \\ 0 & 0.2083 & 0.2083 & -0.2083 & 0 & 0 & -0.2083 & 0 & 0 & 0 & 0 & 0 \\ -0.5208 & 0.2083 & 1.4583 & -0.3125 & -0.5208 & 0.1042 & -0.4167 & 0.3125 & 0 & -0.3125 & 0 & 0 \\ 0.1042 & -0.2083 & -0.3125 & 1.4583 & 0.2083 & -0.2083 & 0.3125 & -1.0417 & -0.3125 & 0 & 0 & 0 \\ 0 & 0 & -0.5208 & 0.2083 & 0.7292 & -0.3125 & 0 & 0 & -0.2083 & 0.1042 & 0 & 0 \\ 0 & 0 & 0.1042 & -0.2083 & -0.3125 & 0.7292 & 0 & 0 & 0.2083 & -0.5208 & 0 & 0 \\ 0 & -0.2083 & -0.4167 & 0.3125 & 0 & 0 & 1.4583 & -0.3125 & -1.0417 & 0.3125 & 0 & -0.1042 \\ -0.1042 & 0 & 0.3125 & -1.0417 & 0 & 0 & -0.3125 & 1.4583 & 0.3125 & -0.4167 & -0.2083 & 0 \\ 0 & 0 & 0 & -0.3125 & -0.2083 & 0.2083 & -1.0417 & 0.3125 & 1.4583 & -0.3125 & -0.2083 & 0.1042 \\ 0 & 0 & -0.3125 & 0 & 0.1042 & -0.5208 & 0.3125 & -0.4167 & -0.3125 & 1.4583 & 0.2083 & -0.5208 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.2083 & -0.2083 & 0.2083 & 0.2083 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.1042 & 0 & 0.1042 & -0.5208 & 0 & 0.5208 \end{bmatrix}$$

The above system can be solved in the usual way by eliminating the rows and columns corresponding to the prescribed displacements $u_1, v_1, u_2, v_2, u_3, v_3, u_5, u_6$ and v_6:

$$10^{10} \begin{bmatrix} 1.4583 & -0.3125 & 0.3125 \\ -0.3125 & 1.4583 & -0.4167 \\ 0.3125 & -0.4167 & 1.4583 \end{bmatrix} \begin{Bmatrix} u_4 \\ v_4 \\ v_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1125 \\ -1125 \end{Bmatrix} -(-\delta)10^{10} \begin{Bmatrix} -0.1042 \\ 0 \\ -0.5208 \end{Bmatrix}$$

$$\begin{Bmatrix} u_4 \\ v_4 \\ v_5 \end{Bmatrix} = \begin{Bmatrix} -0.00013 \\ -0.0011 \\ -0.0039 \end{Bmatrix} m$$

Once the nodal displacements have been obtained the corresponding reactions r_{1x}, r_{1y}, r_{2x}, r_{2y}, r_{3x}, r_{3y}, r_{5x}, r_{6x} and r_{6y} can be computed.

$$r = \begin{Bmatrix} r_{1x} \\ r_{1y} \\ r_{2x} \\ r_{2y} \\ r_{3x} \\ r_{3y} \\ r_{4x} \\ r_{4y} \\ r_{5x} \\ r_{5y} \\ r_{6x} \\ r_{6y} \end{Bmatrix} = 10^7 \begin{Bmatrix} 0.118 \\ 0.267 \\ 0.908 \\ 1.1398 \\ -0.403 \\ 2.0144 \\ 0 \\ 0 \\ -0.0534 \\ 0 \\ -0.5698 \\ -3.1806 \end{Bmatrix} N/m^2$$