FINITE Element Method

Homework - 2

AJAY SINGH NEHRA

Solution:-

Strong form:-

$$b_x + \frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \quad - (1)$$

$$b_y + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} = 0 \quad - (2)$$

Boundry conditions :-

from figure

$$u_1 = u_2 = u_3 = V_1 = V_2 = V_3 = 0$$

(fixed nodes 1,2,3)

$u_5 = 0$ (due to symmetry)

$v_6 = \delta = 10^{-2} = 0.01\,m$

Nodal coordinates (x)

Nodes	x	y
1	-3	0
2	-1.5	0
3	0	0
4	-1.5	1.5
5	0	1.5
6	0	3

Connectivity matrix (T)

nodes / elements	1	2	3
1	2	4	1
2	4	2	5
3	3	5	2
4	5	6	4

→ Description of mesh :-

To find descritization of displacement field, we have

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3$$

linear displacement field is defined by three nodes of the triangular mesh and can be written as,

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y$$

$$v = \alpha_4 + \alpha_5 x + \alpha_6 y$$

deriving shape functions from 'u' alone

$$u_1 = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1$$

$$u_2 = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2$$

$$u_3 = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3$$

$$N_i = \frac{1}{2A^{(e)}} (a_i + b_i x + c_i y) \quad - \quad (3)$$

where,

$$a_i = x_j x_k - x_k x_j$$

$b_i = y_j - y_k$

$c_i = x_k - x_j$

stiffness matrix is given by

$$K^{(e)} = \iint_{A^e} B^T D B \, t \, dA \quad - \quad (4)$$

equivallent nodal force vector

$$f^{(e)} = f_\epsilon^{(e)} + f_\sigma^{(e)} + f_b^{(e)} + f_t^{(e)} \quad - \quad (5)$$

$$f_\epsilon^{(e)} = \iint_{A^{(e)}} B^T D \epsilon^0 \, t \, dA$$

$$f_\sigma^{(e)} = \iint_{A^{(e)}} B^T D \sigma^0 \, t \, dA$$

$$f_{(b)}^{(e)} = \iint_{A^{(e)}} N^T b \, t \, dA$$

$$f_t^{(e)} = \iint_{A^{(e)}} N^T t \, t \, dA$$

now, eq (4) can also be written as

$$K_{ij} = \left(\frac{t}{4A}\right)^{(e)} \begin{bmatrix} b_i b_j d_{11} + c_i c_j d_{33} & b_i c_j d_{12} + b_j c_i d_{33} \\ c_i b_j d_{21} + b_i c_j d_{33} & b_i b_j d_{33} + c_i c_j d_{22} \end{bmatrix}$$

$$\text{———} \quad (6)$$

$$\sigma = D \epsilon$$

Scanned by CamScanner

D is constitutive matrix

$$D = \begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{21} & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}$$

$$d_{11} = d_{22} = \frac{E}{(1-\nu^2)} = \frac{10\text{ GPa}}{(1-0.2^2)} = 10.417\text{ GPa}$$

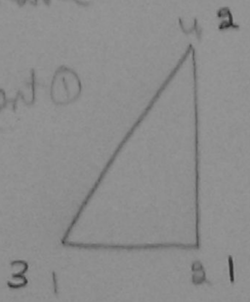
$$d_{12} = d_{21} = \nu d_{11} = 0.2 \times (10.417) = 2.083\text{ GPa}$$

$$d_{33} = \frac{E}{2(1+\nu)} = \frac{10}{2(1+0.2)} = 4.167\text{ GPa}$$

→ computing stiffness matrix for elements 1, 3 & 4

from nodal coordinates and considering local numbering, we have

element ①



1, 2, 4 ⇒ Global numbering

1, 2, 3 ⇒ local numbering

element ① :

$$(x_1, y_1)' = (-1.5, 0)$$

$$(x_2, y_2)' = (-1.5, 1.5)$$

$$(x_3, y_3)' = (-3, 0)$$

But,

$$b_i = y_j - y_k \text{ and } c_i = x_k - x_j$$

$$b_1 = y_2 - y_3 = 1.5 \;;\; c_1 = x_3 - x_2 = -1.5$$

$$b_2 = y_3 - y_1 = 0 \;;\; c_2 = x_1 - x_3 = 1.5$$

$$b_3 = y_1 - y_2 = -1.5 \;;\; c_3 = x_2 - x_1 = 0$$

using eq^n 6 :-

$$K_{ij} = \left(\frac{t}{4A}\right)^e \begin{bmatrix} b_i b_j d_{11} + c_i c_j d_{33} & b_i c_j d_{12} + b_j c_i d_{33} \\ c_i b_j d_{21} + b_i c_j d_{33} & b_i b_j d_{33} + c_i c_j d_{22} \end{bmatrix}$$

$$K_{11}^{1} = \frac{2}{9} \begin{bmatrix} 2.25 \times 10.417 + 2.25 \times 4.167 & -2.25 \times 2.083 - 2.25 \times 4.167 \\ -2.25 \times 2.083 + (-2.25) \times 4.167 & 2.25 \times 10.417 + 2.25 \times 4.167 \end{bmatrix}$$

$$K_{11}^{1} = \frac{2 \times 2.25}{9} \begin{bmatrix} 10.417 + 4.167 & -2.083 - 4.167 \\ -2.083 - 4.167 & 10.417 + 4.167 \end{bmatrix}$$

$$K_{11}^{1} = \begin{bmatrix} 7.292 & -3.125 \\ -3.125 & 7.292 \end{bmatrix}$$

$$K_{12}^{1} = \frac{2 \times 2.25}{9} \begin{bmatrix} -4.167 & 2.083 \\ 4.167 & -10.417 \end{bmatrix} = \begin{bmatrix} -2.083 & 1.042 \\ 2.083 & -5.2085 \end{bmatrix}$$

$$= K_{21}^{(e)}$$

$$K_{13}^{1} = \frac{2 \times 2.25}{9} \begin{bmatrix} -10.417 & 4.167 \\ 2.083 & -4.167 \end{bmatrix} = \begin{bmatrix} -5.2085 & 2.083 \\ 1.042 & -2.083 \end{bmatrix}$$

$$= K_{31}^{(e)}$$

$$K_{22}^{1} = \frac{2 \times 2.25}{9} \begin{bmatrix} 4.167 & 0 \\ 0 & 10.417 \end{bmatrix} = \begin{bmatrix} 2.083 & 0 \\ 0 & 5.2085 \end{bmatrix}$$

$$K_{23}^{1} = \frac{2 \times 2.25}{9} \begin{bmatrix} 0 & -4.167 \\ -2.083 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2.083 \\ -1.042 & 0 \end{bmatrix}$$

$$= K_{32}^{(e)}$$

$$K_{33}^{(1)} = \frac{2\times 2.25}{9}\begin{bmatrix} 10.417 & 0 \\ -2.083 & 4.167 \end{bmatrix} = \begin{bmatrix} 5.2085 & 0 \\ 0 & 2.083 \end{bmatrix}$$

Now,

$K_{11}^{(3)} = K_{11}^{(4)} = K_{11}^{(1)}$

$K_{12}^{(3)} = K_{12}^{(4)} = K_{12}^{(1)}$

$K_{13}^{(3)} = K_{13}^{(4)} = K_{13}^{(1)}$

$K_{21}^{(3)} = K_{21}^{(4)} = K_{21}^{(1)}$

$K_{22}^{(3)} = K_{22}^{(4)} = K_{22}^{(1)}$

$K_{23}^{(3)} = K_{23}^{(4)} = K_{23}^{(1)}$

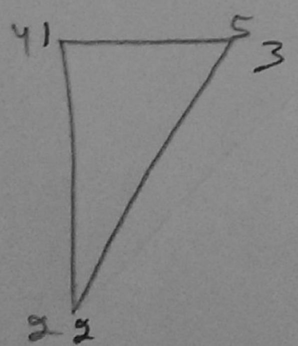
$K_{31}^{(3)} = K_{31}^{(4)} = K_{31}^{(1)}$

$K_{32}^{(3)} = K_{32}^{(4)} \quad K_{32}^{(1)}$

$K_{33}^{(3)} = K_{33}^{(4)} \quad K_{33}^{(1)}$

$\therefore$ element 1, 3 & 4 will have same stiffness matrix

$\Rightarrow$ stiffness matrix for element 2 :



2, 4, 5 $\Rightarrow$ global numbering
1, 2, 3 local numbering

element 2 :-

$$(x_1, y_1)^2 = (-1.5, \ 1.5)$$
$$(x_2, y_2)^2 = (-1.5, \ 0)$$
$$(x_3, y_3)^2 = (0, \ 1.5)$$

$b_i = y_j - y_k$ & $c_i = x_k - x_j$

$b_1 = y_2 - y_3 = -1.5$ $\quad c_1 = x_3 - x_2 = 1.5$

$b_2 = y_3 - y_1 = 0$ $\quad c_2 = x_1 - x_3 = -1.5$

$b_3 = y_1 - y_2 = 1.5$ $\quad c_3 = x_2 - x_1 = 0$

now, element 2 has same values as of element 1, but with opposite sign

$$\therefore K^1 = K^2 = K^3 = K^4$$

$$\underbrace{\begin{bmatrix} K_{11}^1 & K_{12}^1 & K_{13}^1 \\ K_{21}^1 & K_{22}^1 & K_{23}^1 \\ K_{31}^1 & K_{32}^1 & K_{33}^1 \end{bmatrix}}_{K^1} = \underbrace{\begin{bmatrix} K_{11}^2 & K_{12}^2 & K_{13}^2 \\ K_{21}^2 & K_{22}^2 & K_{23}^2 \\ K_{31}^2 & K_{32}^2 & K_{33}^2 \end{bmatrix}}_{K^2}$$

$$\parallel \qquad\qquad \parallel$$

$$\underbrace{\begin{bmatrix} K_{11}^4 & K_{12}^4 & K_{13}^4 \\ K_{21}^4 & K_{22}^4 & K_{23}^4 \\ K_{31}^4 & K_{32}^4 & K_{33}^4 \end{bmatrix}}_{K^4} = \underbrace{\begin{bmatrix} K_{11}^3 & K_{12}^3 & K_{13}^3 \\ K_{21}^3 & K_{22}^3 & K_{23}^3 \\ K_{31}^3 & K_{32}^3 & K_{33}^3 \end{bmatrix}}_{K^3}$$

$$K = \begin{bmatrix} K_{33}^{(1)} & K_{13}^{(1)T} & 0 & K_{23}^{(1)} & 0 & 0 \\ & K_{11}^{(1)} + K_{22}^{(2)} + K_{33}^{(3)} & K_{13}^{(3)T} & K_{12}^{(1)} + K_{13}^{(2)T} & K_{23}^{(2)} + K_{25}^{(3)T} & 0 \\ & & & & K_{12}^{(3)} & 0 \\ & & K_{11}^{(3)} & 0 & & \\ & & & K_{22}^{(1)} + K_{11}^{(2)} + K_{33}^{(4)} & K_{13}^{(2)} + K_{13}^{(4)T} & K_{23}^{(4)T} \\ & & & & K_{11}^{(4)} + K_{22}^{(3)} + K_{22}^{(2)} & K_{12}^{(4)} \\ & & & & & K_{22}^{(4)} \end{bmatrix}$$

we know that,

$$a_i^{(e)} = \begin{bmatrix} u_i \\ v_i \end{bmatrix} \Rightarrow a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

for this problem, displacement matrix 'a' is

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}$$

and nodal force vector 'f', is

$$f = \begin{bmatrix} f_3^{(1)} + r_1 \\ f_1^{(1)} + f_2^{(2)} + f_3^{(3)} + r_2 \\ f_1^{(3)} + r_3 \\ f_2^{(1)} + f_1^{(2)} + f_3^{(4)} \\ f_3^{(2)} + f_2^{(3)} + f_1^{(4)} \\ f_2^{(4)} \end{bmatrix}$$

$$Ka = f$$

This problem have 12 degrees of freedom, where 9 degrees are contrained

computing nodal displacements:-

from boundry conditions

$$u_1 = u_2 = u_3 = v_1 = v_2 = v_3 = 0 ,$$

so $\quad a_1 = a_2 = a_3 = 0$

so we have to consider rows 4, 5 & 6 for nodal global stiffness matrix

and $\quad u_5 = u_6 = 0$

$$\begin{bmatrix} K_{22}^{(1)} + K_{11}^{(2)} + K_{33}^{(4)} & \left(K_{13}^{(2)} + K_{13}^{(4)}\right)^T & K_{23}^{(4)T} \\ & K_{11}^{(4)} + K_{22}^{(3)} + K_{33}^{(2)} & K_{12}^{(4)} \\ & & K_{22}^{(4)} \end{bmatrix} \begin{bmatrix} u_4 \\ v_4 \\ u_5 = 0 \\ v_5 \\ u_6 = 0 \\ v_6 = \delta = 10^{-2} m \end{bmatrix} = \begin{bmatrix} f_2^{(1)} + f_1^{(2)} + f_3^{(4)} \\ f_3^{(2)} + f_2^{(3)} + f_1^{(4)} \\ f_2^{(4)} \end{bmatrix} \quad \text{— (7)}$$

now substituting value of $K_{ij}^{(e)}$ #

$$K = \begin{bmatrix} -14.5835 & -3.125 & -10.417 & 3.125 & 0 & 1.042 \\ -3.125 & 14.5835 & 4.166 & -4.166 & -1.042 & 0 \\ -10.417 & 4.166 & 14.5835 & -3.125 & -2.083 & 1.042 \\ 3.125 & -4.166 & -3.125 & 14.5835 & 2.083 & -5.2085 \\ 0 & -1.042 & -2.083 & 2.083 & 2.083 & 0 \\ 1.042 & 0 & 1.042 & -5.2085 & 0 & 5.2085 \end{bmatrix} \text{GN/m}$$

we know that the deformation in domain is due to self-weight and gravity is acting in the direction of y-axis.

∴ only body forces are significant for consideration and there are no surface loads.

Body forces for equivalent force

$$f_{bi} = \left(\frac{At}{3}\right)^{(e)} \begin{bmatrix} b_x \\ b_y \end{bmatrix} \quad — \quad (8)$$

but,

$$b_x = 0$$

$$b_y = -\rho g = -10^3$$

$$f_{bi} = \left(\frac{2.25}{6}\right) \begin{bmatrix} 0 \\ -10^3 \end{bmatrix} = \begin{bmatrix} 0 \\ -375 \end{bmatrix} \text{N}$$

substituting $f_{bi}^{(e)}$ and K in eq^n (7)

$$\begin{bmatrix} 14.5835 & -3.125 & 3.125 & 1.042 \\ -3.125 & 14.5835 & -4.166 & 0 \\ 3.125 & -4.166 & 14.5835 & -5.2085 \\ 1.042 & 0 & -5.2085 & 5.2085 \end{bmatrix} \times 10^9 \begin{bmatrix} u_4 \\ v_4 \\ v_5 \\ -10^{-2} \end{bmatrix} = \begin{bmatrix} 0 \\ -1125 \\ -1125 \\ -375 \end{bmatrix}$$

By solving the system of equations

nodal displacements u_4, v_4, v_5 are

$$\boxed{\begin{aligned} u_4 &= -1.99 \times 10^{-4} \\ v_4 &= -1.13 \times 10^{-3} \\ v_5 &= -3.87 \times 10^{-3} \end{aligned}}$$