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FINITE ELEMENT – HOMEWORK 1

FINITE ELEMENTS Homework # 1

Basics of FE *Master in Computational Mechanics*

Consider the following differential equation

$$-u'' = f \text{ in }]0,1[$$

with the boundary conditions $u(0) = 0$ and $u(1) = \alpha$.

The Finite Element discretization is a 2-noded linear mesh given by the nodes $x_i = ih$ for $i = 0, 1, \ldots, n$ and $h = 1/n$.

1. Find the weak form of the problem. Describe the FE approximation u^h.
2. Describe the linear system of equations to be solved.
3. Compute the FE approximation u^h for $n = 3$, $f(x) = \sin x$ and $\alpha = 3$. Compare it with the exact solution, $u(x) = \sin x + (3 - \sin 1)x$.

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1) The weak form of the problem:

We have:

$$-\frac{d^2u}{dx^2} = f$$

Or

$$\frac{d^2u}{dx^2} + f = 0$$

In the interval [0,1]

$$A(u) = \frac{d}{dx}\left(\frac{du}{dx}\right) + f(x) = 0$$

And the boundary conditions (BC) are:

$$B(u) \quad \begin{cases} u(0) = 0 \\ u(1) = \alpha \end{cases}$$

The unknown u is approximated by

$$u = u^h = \sum_{i=1}^{n} Ni(x)ai$$

$$\frac{du}{dx} = \frac{dNi}{dx}$$

Polynomial approximation of the unknown

$$u(x) \approx \hat{u}\,(x) = 1 + \sum_{j=1}^{n} a_j\, x^i$$

$$u(x) = 1 + a_1 x + a_2 x^2$$

Shape functions N1

$$\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

1. The weak form of the problem is:

$$\int_0^1 w \frac{d^2u}{dx^2} dx = \int_0^1 wif dx$$

And integrating by parts we have:

$$\int_0^1 \left\{ \left[\frac{\partial^2 u}{\partial x^2} \right] \cdot W \right\} dx = W \frac{\partial u}{\partial x}\bigg|_0^1 + \int_0^1 \frac{\partial W}{\partial x} \frac{\partial u}{\partial x} dx$$

$$\int_0^1 \left\{ \left[\frac{\partial^2 u}{\partial x^2} \right] \cdot W \right\} dx = \int_0^1 \frac{\partial W}{\partial x} \frac{\partial u}{\partial x} dx$$

$$\int_0^1 W \cdot f dx = \int_0^1 \frac{\partial W}{\partial x} \frac{\partial u}{\partial x} dx$$

$$U(x) = \sum_{i=0}^{1} N_i \cdot u_i$$

$$W(x) = N_i(\mathrm{x})$$

And the weak form of the problem reads:

$$\int_0^1 \frac{\partial W}{\partial x} \frac{\partial u}{\partial x} dx = \int_0^1 W \cdot f dx + \mathrm{w}. \frac{\partial u}{\partial x}\bigg|_0^1$$

And the weighted residual formulation

$$\int_0^1 \frac{dwi}{dx}\frac{d\hat{u}}{dx}dx = \int_0^1 wifdx + wiq - wi\bar{q}$$

$\uparrow$ in x =0 $\quad$ $\uparrow$ in x = 1

2.

$$\int_0^1 \frac{\partial W_i}{\partial x}\sum_{j=0}^{n}\frac{\partial N_j}{\partial x}a_j\,dx = \int_0^1 W_i \cdot f dx + [W_i.R]_1 + [W_i.R]_0$$

$$W_i = N_i$$

$$\int_0^1 \frac{\partial N_i}{\partial x}\sum_{j=0}^{n}\frac{\partial N_j}{\partial x}a_j\,dx = \int_0^1 N_i \cdot f dx + [N_i.R]_1 + [N_i.R]_0$$

So the global equation for this problem is:

$$\int_0^1 \frac{\partial W_i}{\partial x}\left(\frac{\partial N_1}{\partial x}a_1 + \frac{\partial N_2}{\partial x}a_2 + \frac{\partial N_3}{\partial x}a_3 + \frac{\partial N_n}{\partial x}a_4\right)dx = \int_0^1 W_1 \cdot f dx + [N_i.R]_1 + [N_i.R]_0$$

For I =1

$$\int_0^{1/3} \frac{\partial N_1{}^{(1)}}{\partial x}\left(\frac{\partial N_1{}^{(1)}}{\partial x}a_1 + \frac{\partial N_2{}^{(1)}}{\partial x}a_2\right)dx = \int_0^{1/3} N_1{}^{(1)} \cdot f dx + R_0$$

For I= 2

$$\int_0^{1/3} \frac{\partial N_2{}^{(1)}}{\partial x}\left(\frac{\partial N_1{}^{(1)}}{\partial x}a_1 + \frac{\partial N_2{}^{(1)}}{\partial x}a_2\right)dx + \int_{1/3}^{2/3} \frac{\partial N_1{}^{(2)}}{\partial x}\left(\frac{\partial N_1{}^{(2)}}{\partial x}a_1 + \frac{\partial N_2{}^{(2)}}{\partial x}a_2\right)dx = \int_{1/3}^{2/3} N_1{}^{(1)} \cdot f dx + \int_0^{1/3} N_1{}^{(2)} \cdot f dx$$

For I = 3

$$\int_{1/3}^{2/3} \frac{\partial N_2{}^{(2)}}{\partial x}\left(\frac{\partial N_1{}^{(2)}}{\partial x}a_1 + \frac{\partial N_2{}^{(2)}}{\partial x}a_2\right)dx + \int_{1/3}^{2/3} \frac{\partial N_1{}^{(3)}}{\partial x}\left(\frac{\partial N_1{}^{(3)}}{\partial x}a_1 + \frac{\partial N_2{}^{(3)}}{\partial x}a_2\right)dx = \int_{1/3}^{2/3} N_1{}^{(1)} \cdot f dx + \int_0^{1/3} N_1{}^{(2)} \cdot f dx$$

For I = 4

$$\int_{2/3}^{1} \frac{\partial N_2{}^{(3)}}{\partial x}\left(\frac{\partial N_1{}^{(3)}}{\partial x}a_1 + \frac{\partial N_2{}^{(3)}}{\partial x}a_2\right)dx = \int_{2/3}^{1} N_2{}^{(3)} \cdot f dx + R_1$$

All this can be written and copmpute as:

K·a = f

And we have:

$$\begin{pmatrix} k11(1) & k12(1) & 0 & 0 \\ k21(1) & k22(1)+k11(2) & k12(2) & 0 \\ 0 & k21(2) & k22(2)+k11(3) & k12(3) \\ 0 & 0 & K21(3) & k22(3) \end{pmatrix} \begin{Bmatrix} u1 \\ u2 \\ u3 \\ u4 \end{Bmatrix} = \begin{Bmatrix} f+q0 \\ f+f2 \\ f+f3 \\ f-q \end{Bmatrix}$$

General expressions:

$$K_{ij} = \int_0^1 \frac{dNi}{dx}\frac{dNj}{dx}dx$$

2-noded linear element:

$$\frac{dN1}{dx} = \frac{-1}{h^{(e)}}$$

$$\frac{dN2}{dx} = \frac{1}{h^{(e)}}$$

$$N_1^{(e)} = \frac{x_2^{(e)} - x}{h^{(e)}}$$

$$N_2^{(e)} = \frac{x - x_1^{(e)}}{h^{(e)}}$$

In our problem we have n=3 and h = 1/n =1/3

$$k_{11}^{(e)} = k_{22}^{(e)} = \int_0^{1/3} \frac{dN1}{dx}\frac{dN1}{dx}dx = \int_0^{1/3} \frac{1}{\frac{-1}{3}} \cdot \frac{1}{\frac{-1}{3}}dx = 9x = 3$$

↓ For 0 to 1/3

$$k_{21}^{(e)} = k_{12}^{(e)} = \int_0^{1/3} \frac{dN2}{dx}\frac{dN2}{dx}dx = \int_0^{1/3} \frac{1}{\frac{-1}{3}} \cdot \frac{1}{\frac{1}{3}}dx = -9x = -3$$

↓ For 0 to 1/3

The value of k is the same for all the elements that I have because h is the same for the 3 elements.

3. The function given:

$$f(x) = \sin(x)$$

$$\alpha = 3$$

$f_1^{(1)} = \int_0^{1/3} N_1^{(1)} \cdot f(x)dx = \int_0^{1/3} \frac{x_2^{(1)}-x}{h^{(1)}} \cdot \sin(x)\,dx = \int_0^{1/3} \frac{x_2^{(1)}}{h^{(1)}} \cdot \sin(x)\,dx + \int_0^{1/3} \frac{-x}{h^{(1)}} \cdot \sin(x)\,dx =$
$\underbrace{\frac{-x_2^{(1)}}{h^{(1)}} \cdot \cos(x)}_{\text{For 0 to 1/3}} - \underbrace{\frac{1}{h^{(1)}} \cdot (\sin(x) - x \cdot \cos(x))}_{\text{For 0 to 1/3}}$ = -1-3·sin(1/3) = 0.0184

$f_2^{(1)} = \int_0^{1/3} N_2^{(1)} \cdot f(x)dx = \int_0^{1/3} \frac{x-x_1^{(1)}}{h^{(1)}} \cdot \sin(x)\,dx$ = 3(sin(1/3)-(1/3)·cos(1/3))= 3·sin(1/3) – cos(1/3) = 0.0366

$f_1^{(2)} = \int_{1/3}^{2/3} N_1^{(2)} \cdot f(x)dx = \int_{1/3}^{2/3} \frac{x3-x}{h} \cdot \sin(x)\,dx = -\frac{x3}{h} \cdot \cos(x) - \frac{1}{h} \cdot (\sin(x) - x \cdot \cos(x))$= 2 cos (1/3) -3 sin(2/3) + 3 sin(1/3) – cos (1/3) = 0.0714

$f_2^{(2)} = \int_{1/3}^{2/3} N_1^{(2)} \cdot f(x)dx = \int_{1/3}^{2/3} \frac{-x2}{h} \cdot \sin(x)\,dx + \int_{1/3}^{2/3} \frac{x}{h} \cdot \sin(x)\,dx = -\frac{x2}{h} \cdot \cos(x) + \frac{1}{h} \cdot (\sin(x) - x \cdot \cos x)$= cos(2/3) + 3·sin(2/3) – 3·sin(1/3) = 0.0877

$f_1^{(3)} = \int_{2/3}^{1} \frac{x4}{h} \cdot \sin(x)\,dx + \int_{2/3}^{1} \frac{-x}{h} \cdot \sin(x)\,dx = -\frac{x4}{h} \cdot \cos(x) - \frac{1}{h} \cdot (\sin(x) - x \cdot \cos(x))$ = cos (2/3) -3sin(1) +3sin(2/3) = 0.1163

$f_2^{(3)} = \int_{2/3}^{1} \frac{-x3}{h} \cdot \sin(x)\,dx + \int_{2/3}^{1} \frac{x}{h} \cdot \sin(x)\,dx = \frac{x3}{h} \cdot \cos(x) + \frac{1}{h} \cdot (\sin(x) - x \cdot \cos(x))$= -cos(1) + 3sin(1) -3sin(2/3) = 0.1291

Then the matrix reads:

$$3\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ u \\ v \\ 3 \end{bmatrix} = \begin{bmatrix} 0.0184 + c \\ 0.108 \\ 0.2042 \\ 0.1291 + d \end{bmatrix}$$

Where

R0 = c ; R1 = d ; u = a2 ; v = a3

We solve this and we obtaine:

$9a_3$ = 18.5164

a_3 = 2.0573

$6a_2 = 0.108 + 3a_3$

a_2 = 1.0466

For the exact solution we have:

$$u(x) = \sin(x) + \ 3(\sin(1))x$$

And with this :

u(0) = 1.0467

u(1/3) = 1.0467

u(2/3) = 2.0573

u(1) = 3

so the FEM solution is equal to the exact solution.