Finite Elements
Home Work 2
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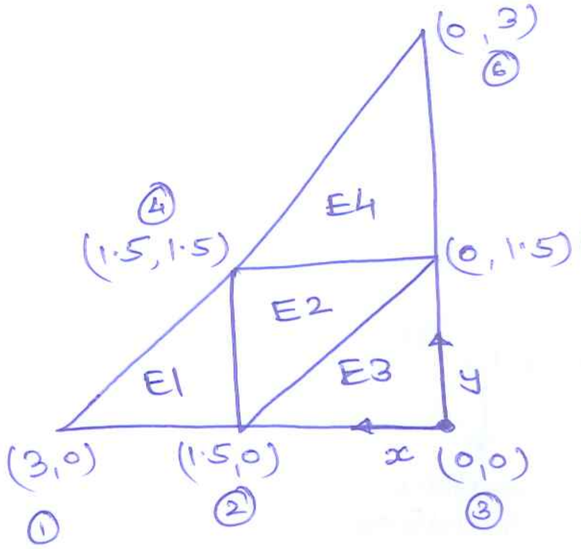


The adjacent diagram shows the chosen global coordinate system. (GCS) E1, E2, E3, E4 are the elements nomenclatures. [1 2 3 ....] are the global node numbers

The differential equations that we are solving in this problem, are as follows -

(A) Equations of equilibrium

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + f_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + f_y = 0$$

f_x, f_y are body forces in x & y directions

(B) Constitutive Equations.

$$\sigma = D\epsilon$$

$$\sigma = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} \qquad \epsilon = \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

since plane stress problem, and material is given tobe isotropic.

$$D = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

E = Young's Modulus.

ν = Poisson's Ratio

(C) Kinematic Equations

$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix}$$

The weak form of above set of equations can be obtained using either Weighted Residual Method or Principle of Virtual Work.

Weak Form Description ⇒

For every element (e)

$$K^{(e)} a^{(e)} - f^{(e)} = q^{(e)}$$

$$K^{(e)} = \iint_{A^{(e)}} B^T D B \, t \, dA \qquad f^{(e)} = \underbrace{f_b^{(e)}}_{\text{Body Forces}} + \underbrace{f_t^{(e)}}_{\text{surface Tractions}}$$

$q^{(e)}$ = External Nodal Forces.

$$\sum_e q_i^{(e)} = \underbrace{P_j}_{\text{External Forces on the system.}}$$

Note; Even though, there will be $f_t^{(e)}$ and $q^{(e)}$ on every element face, node; we are only concerned about the external forces and surface tractions since when the forces/tractions are assembled, internal forces/tractions will cancel each other due to Newton's Third Law. te

The definitions of some quantities in the above equations,

$$B = [B_1 \;\; B_2 \;\; B_3]$$

$$B_i = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} \end{bmatrix}$$

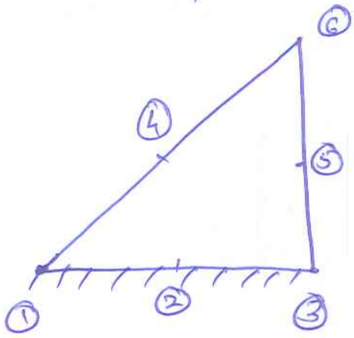
Every node has 2-DOFs. All triangles

$$\left.\begin{aligned} u &= N_1 u_1 + N_2 u_2 + N_3 u_3 \\ v &= N_1 v_1 + N_2 v_2 + N_3 v_3 \end{aligned}\right] \text{elemental}$$

$$u = N a \qquad \overline{N_i} = \begin{bmatrix} N_i & 0 \\ 0 & N_i \end{bmatrix}$$

$$N = [\overline{N_1} \;\; \overline{N_2} \;\; \overline{N_3}] \qquad a = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

Boundary conditions.

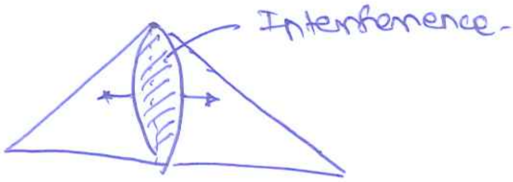


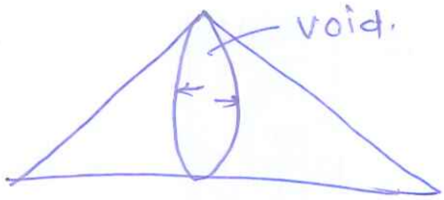
since the plate is fixed at the bottom, ~~the~~ the line at the bottom can't move in any direction.

$$\boxed{\begin{array}{ccc} u_1 = 0 & u_2 = 0 & u_3 = 0 \\ V_1 = 0 & V_2 = 0 & V_3 = 0 \end{array}}$$

(Also on full line)

At the line 3-5-6, there is a symmetry, so if the point next to this line moves rightward, there would be interference and in other case it would create void.



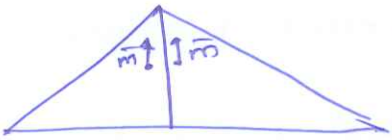


Therefore, ~~the~~ on the plane of symmetry, the displacement perpendicular to the plane should be zero.

$$\Rightarrow \quad \boxed{u_6 = 0 \qquad u_5 = 0 \qquad u_3 = 0}$$

(Also on full line)

Also, if the point on left of line moves vertically by $\overline{m}$ then the point on right of line should move vertically by $\overline{m}$ (due to symmetry)



$\Rightarrow$ Across the boundary the variation of v is zero. Mathematically,

$$\frac{\partial v}{\partial x} = 0$$

so cumulatively, on the central vertical line,

$$u = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial y} = 0 \quad (\text{since it is zero (constant) along } y)$$

$$\Rightarrow \quad \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0 \quad \Rightarrow \quad \varepsilon_{xy} = 0 \quad \Rightarrow \quad \boxed{\tau_{xy} = 0}$$

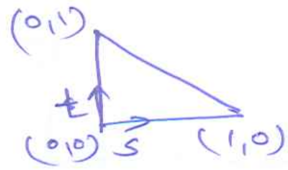
Total unknowns,

$u_1 = 0$	$\boxed{u_4 = ?}$	$V_1 = 0$	$\boxed{V_4 = ?}$
$u_2 = 0$	$u_5 = 0$	$V_2 = 0$	$\boxed{V_5 = ?}$
$u_3 = 0$	$u_6 = 0$	$V_3 = 0$	$\boxed{V_6 = ?}$

Applied displacement ~~given as~~ load.

since K is to be found out (an integration) it ~~is~~ seems easy to me to do the integration in isoparametric domain. ④

$$\underbrace{\iint \, dx\,dy}_{\text{ACTUAL}} = \underbrace{\iint |J| \, ds\,dt}_{\text{ISOPARAMETRIC}}$$



$|J|$ is the determinant of Jacobian Matrix.

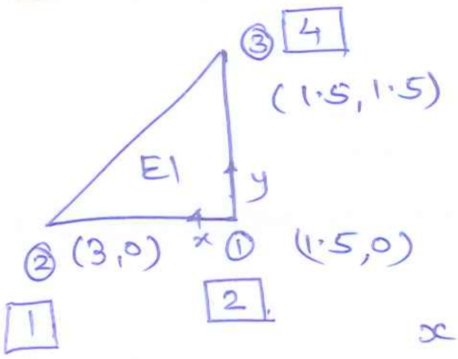
$$\begin{bmatrix} \frac{\partial N_i}{\partial s} \\ \frac{\partial N_i}{\partial t} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{bmatrix} \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial N_i}{\partial s} \\ \frac{\partial N_i}{\partial t} \end{bmatrix}$$

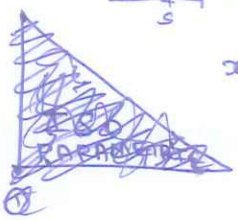
Note: The motivation behind using isoparametric elemental integration is that the shape functions have simple form in isoparametric form, so easy to differentiate and integrate.

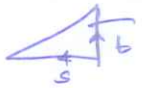
lets work out the connectivities and calculate other quantities like J & J^{-1} etc.

circled nos are local node numbers $\boxed{}$ are global numbers.

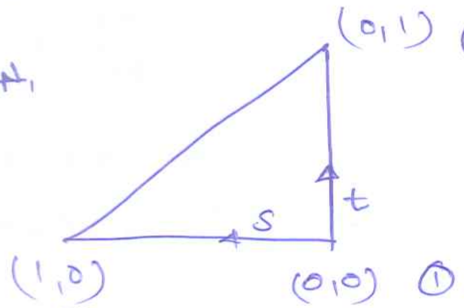
Element 1







~~x = N1~~



$N_1^{(e)} = 1-s-t$

$N_2^{(e)} = s$

$N_3^{(e)} = t$

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3$$

$$x = (1-s-t)(1.5) + s(3) + t(1.5)$$

$$= 1.5 + 1.5 s$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3$$

$$= (1-s-t)(0) + s(0) + t(1.5)$$

$$= 1.5 t$$

$$J = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{bmatrix} = \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix} \qquad J^{-1} = \begin{bmatrix} 2/3 & 0 \\ 0 & 2/3 \end{bmatrix}$$

$$|J| = 2.25$$

$$N = \left[\begin{array}{cc|cc|cc} 1-s-t & 0 & s & 0 & t & 0 \\ 0 & 1-s-t & 0 & s & 0 & t \end{array}\right]$$

$$\begin{bmatrix} \frac{\partial N_1}{\partial x} \\ \frac{\partial N_1}{\partial y} \end{bmatrix} = \begin{bmatrix} 2/3 & 0 \\ 0 & 2/3 \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial s} \\ \frac{\partial N_1}{\partial t} \end{bmatrix} = \begin{bmatrix} 2/3 & 0 \\ 0 & 2/3 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2/3 \\ -2/3 \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial N_2}{\partial x} \\ \frac{\partial N_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 2/3 & 0 \\ 0 & 2/3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial N_3}{\partial x} \\ \frac{\partial N_3}{\partial y} \end{bmatrix} = \begin{bmatrix} 2/3 & 0 \\ 0 & 2/3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2/3 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} \end{bmatrix} = \begin{bmatrix} -2/3 & 0 \\ 0 & -2/3 \\ -2/3 & -2/3 \end{bmatrix} \qquad B_2 = \begin{bmatrix} 2/3 & 0 \\ 0 & 0 \\ 0 & 2/3 \end{bmatrix}$$

$$B_3 = \begin{bmatrix} 0 & 0 \\ 0 & 2/3 \\ 2/3 & 0 \end{bmatrix} \qquad B = [B_1 \; B_2 \; B_3]$$

$$B = \left[\begin{array}{cc|cc|cc} -2/3 & 0 & 2/3 & 0 & 0 & 0 \\ 0 & -2/3 & 0 & 0 & 0 & 2/3 \\ -2/3 & -2/3 & 0 & 2/3 & 2/3 & 0 \end{array}\right]$$

thickness $= 1$

$$K^{(1)} = \iint_A B^T D B \, dA$$

$$= \int_0^1 \int_0^{1-t} B^T D B \, |J| \, ds\,dt$$

D is constant with respect s, t
B is constant with respect s, t } Take out of Integration.
$|J|$ is constant with respect to s, t

$$= |J| B^T D B \int_0^1 \int_0^{1-t} ds\,dt = \frac{|J|}{2} B^T D B$$

$K^{(1)}$ is a 6×6 matrix

$$\underset{}{\frac{1}{2}} \times \underset{(1\times1)}{|J|} \times \underset{(6\times3)}{B^T} \; \underset{(3\times3)}{D} \; \underset{(3\times6)}{B} = K^{(1)}$$

Now Forces on Element 1

1. Body Force (weight)
2. surface force (but will be distributed on nodes)
3. Nodal Forces (Reaction)

$$F_b^{(e)} = \iint_A N^T b\,t\,dA = \int_0^1 \int_0^{1-t} N^T \begin{bmatrix} 0 \\ -\rho g \end{bmatrix} |J| \, ds\,dt$$

$$N = \left[\begin{array}{cc|cc|cc} 1-s-t & 0 & s & 0 & t & 0 \\ 0 & 1-s-t & 0 & s & 0 & t \end{array}\right]$$

The integration gives the following result.

$$F_b^{(e)} = \begin{bmatrix} 0 \\ -\tfrac{1}{3} \\ 0 \\ -\tfrac{1}{3} \\ 0 \\ -\tfrac{1}{3} \end{bmatrix} \rho g |J| \qquad \begin{matrix} f_1^{(1)} \\ f_2^{(1)} \\ f_3^{(1)} \end{matrix}$$

Surface forces along fixed bottom edge and nodal forces can together be called as reactions we will have two reactions r_1 & r_2 at nodes 1 & 2 (GCS) other nodal forces and surface forces are internal to the system and will get cancelled out when summed.

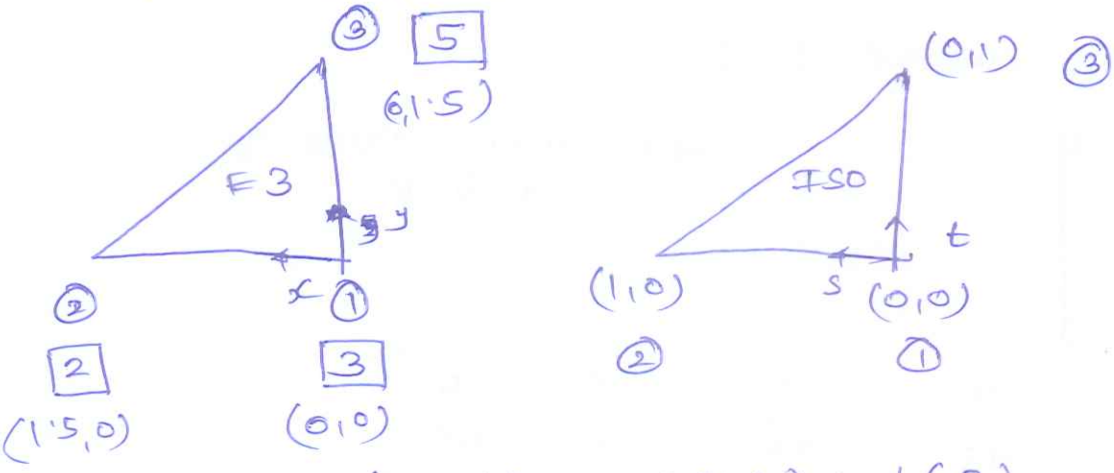
Local equation for element 1 is

with given values of E & ν

$$E \times \begin{bmatrix} \frac{35}{48} & \frac{5}{16} & \frac{-25}{48} & \frac{-5}{24} & \frac{-5}{24} & \frac{-5}{48} \\ & \frac{35}{48} & \frac{-5}{48} & \frac{-5}{24} & \frac{-5}{24} & \frac{-25}{48} \\ & & \frac{25}{48} & 0 & 0 & \frac{5}{48} \\ & & & \frac{5}{24} & \frac{5}{24} & 0 \\ & \text{symmetric} & & & \frac{5}{24} & 0 \\ & & & & & \frac{25}{48} \end{bmatrix} = k_1^{(e)}$$

Element 3

Element 2 will be considered later. Element 3 is just offset of element 1. Expecting some similarity with element 1.



$$x = (1-s-t)\,0 + s(1{\cdot}5) + t(0)$$

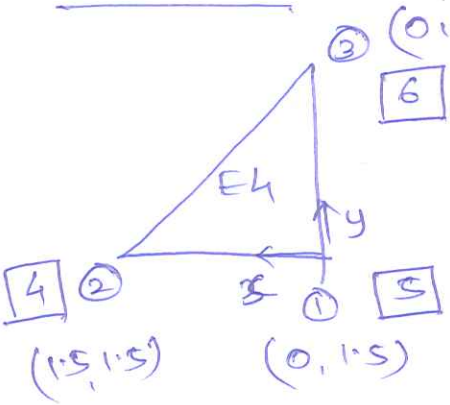
$$= 1{\cdot}5\,s$$

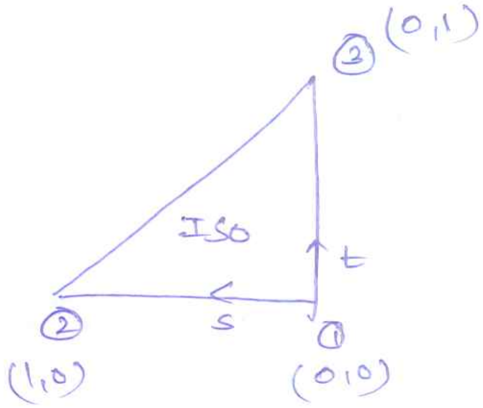
$$y = 1{\cdot}5\,t$$

$$J = \begin{bmatrix} 1{\cdot}5 & 0 \\ 0 & 1{\cdot}5 \end{bmatrix} \xrightarrow{\text{Implies}}$$ same as element 1, $|J|$ also same

Hence
$$B^{(3)} = B^{(1)}$$
$$K^{(3)} = K^{(1)}$$
$$F_b^{(3)} = F_b^{(1)}$$

Element 4





$$x = 1.5\,s$$

$$y = 1.5 + 1.5\,t$$

$$J = \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix}$$ same as element 2

Implies $|J|$ also same.

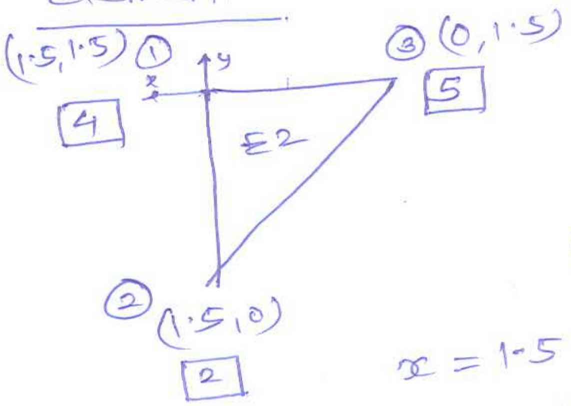
Hence,

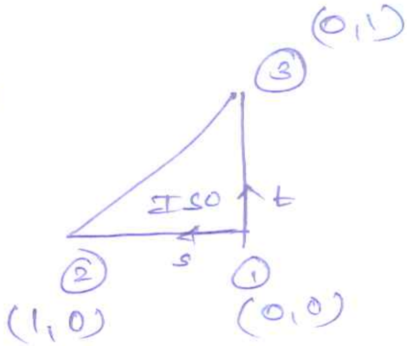
$$B^{(4)} = B^{(1)}$$

$$K^{(4)} = K^{(1)}$$

$$F_b^{(4)} = F_b^{(1)}$$

Element 2





$$x = 1.5 - 1.5\,t$$

$$y = 1.5 - 1.5\,s$$

$$J = \begin{bmatrix} -1.5 & 0 \\ 0 & -1.5 \end{bmatrix}$$ different as element 1

$|J|$ is same though.

$$B = \left[\begin{array}{cc|cc|cc} 2/3 & 0 & -2/3 & 0 & 0 & 0 \\ 0 & 2/3 & 0 & 0 & 0 & -2/3 \\ 2/3 & 2/3 & 0 & -2/3 & -2/3 & 0 \end{array}\right]$$

But since $|J|$ is same,

$$F_b^{(1)} = F_b^{(2)}$$

From B on page 9, it is evident that $B^{(2)} = -B^{(1)}$

B

so $K^{(2)} = \dfrac{B^T D B\, |J|}{2}$

$$K^{(2)} = (B^2)^T D\, B^2 \frac{|J^2|}{2}$$

J^2 is not $J \times J$

$$= \underbrace{(-)B^{(1)\,T}\, D\, (-)B^{(1)}}_{\text{product of two negatives}} \frac{|J^{(1)}|}{2}$$

$|J|$ is same for element 1 & 2

$\Rightarrow$

$$K^{(2)} = K^{(1)}$$

Connectivity Matrix

— local Nos.

	①	②	③
Elem 1	2	1	4
Elem 2	4	5	2
Elem 3	3	2	5
Elem 4	5	4	6

values in table are the ~~designations~~ global node nos.

so for example, First row reads 2 1 4 means Element 1 is made up of 3 nodes

1st node (local no) is global node # 2

2nd node (local no) is global node # 1

so on.

Assembly of global stiffness matrix.

using connectivity matrix, global ~~equations~~ K can be ~~[illegible]~~ written as, * Refer to attached excel sheet or MATLAB CODE to view global stiffness matrix

$K =$

$K_{22}^{(1)}$	$K_{21}^{(1)}$	0	$K_{23}^{(1)}$	0	0
$K_{12}^{(1)}$	$K_{11}^{(1)}$ $+K_{22}^{(3)}$ $+K_{33}^{(2)}$	$K_{21}^{(3)}$	$K_{13}^{(1)}$ $K_{31}^{(1)}$	$K_{32}^{(2)}$ $K_{23}^{(3)}$	0
0	$K_{12}^{(3)}$	$K_{11}^{(3)}$	0	$K_{13}^{(3)}$	0
$K_{32}^{(1)}$	$K_{31}^{(1)}$ $K_{13}^{(2)}$	0	$K_{33}^{(1)}$ $K_{11}^{(2)}$ $K_{22}^{(4)}$	$K_{12}^{(2)}$ $K_{21}^{(4)}$	$K_{23}^{(4)}$
0	$K_{23}^{(2)}$ $K_{32}^{(3)}$	$K_{31}^{(3)}$	$K_{21}^{(2)}$ $K_{12}^{(4)}$	$K_{22}^{(2)}$ $K_{33}^{(3)}$ $K_{11}^{(4)}$	$K_{13}^{(4)}$
0	0	0	$K_{32}^{(4)}$	$K_{31}^{(4)}$	$K_{33}^{(4)}$

* multiple entries in one box indicate addition.

where $K^{(e)}$ is 6×6 matrix and is made into groups of (2×2) ⇒ makes it a 3×3 matrix.

for eg. $K^{(1)}/E =$

$$\begin{bmatrix} \frac{35}{48} & \frac{5}{16} & -\frac{25}{48} & -\frac{5}{24} & -\frac{5}{24} & -\frac{5}{48} \\ \frac{5}{16} & \frac{35}{48} & -\frac{5}{48} & -\frac{5}{24} & -\frac{5}{24} & -\frac{25}{48} \\ -\frac{25}{48} & -\frac{5}{48} & \frac{25}{48} & 0 & 0 & \frac{5}{48} \\ -\frac{5}{24} & -\frac{5}{24} & 0 & \frac{5}{24} & \frac{5}{24} & 0 \\ -\frac{5}{24} & -\frac{5}{24} & 0 & \frac{5}{24} & \frac{5}{24} & 0 \\ -\frac{5}{48} & -\frac{25}{48} & \frac{5}{48} & 0 & 0 & \frac{25}{48} \end{bmatrix}$$

9 circles are the 2x2 groups.

$$K_{11}^{(1)} = \begin{bmatrix} \frac{35}{48} & \frac{5}{16} \\ \frac{5}{16} & \frac{35}{48} \end{bmatrix}$$

$\frac{K_{global}}{E}$ = Refer to the matlab script (attached) for more precise values

& its a 12x12 matrix.

8.5208

E

Assembly of global Force Matrix.

In $Ka - F = q$

$q = 0$ No external applied nodal Force. All are reactions.

$\Rightarrow Ka = F$

$$F = \begin{array}{l} \begin{bmatrix} F^{(1)}_{2} + r_1 \\ F^{(1)}_{1} + F^{(2)}_{2} + F^{(3)}_{3} + r_2 \\ F^{(3)}_{1} + r_3 \\ F^{(2)}_{1} + F^{(4)}_{2} + F^{(3)}_{1} \\ F^{(4)}_{1} + F^{(2)}_{2} + F^{(3)}_{3} + r_4 \\ F^{(4)}_{3} + r_5 \end{bmatrix} \end{array} \begin{array}{l} \text{(Node 1)} \\ \text{(Node 2)} \\ \text{(Node 3)} \\ \text{(Node 4)} \\ \text{(Node 5)} \\ \text{(Node 6)} \end{array}$$

Note:
As explained in previous sections

After calculation this gives,

$$F = \begin{bmatrix} 0 \\ -375 \\ 0 \\ -1125 \\ 0 \\ -375 \\ 0 \\ -1125 \\ 0 \\ -1125 \\ 0 \\ -375 \end{bmatrix} + \text{Reactions}$$

So the global system of equations to be solved becomes following.

$$K \times \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \\ u_6 \\ v_6 \end{bmatrix} = \begin{bmatrix} 0 \\ -375 \\ 0 \\ -1125 \\ 0 \\ -375 \\ 0 \\ -1125 \\ 0 \\ -1125 \\ 0 \\ -375 \end{bmatrix} + \begin{bmatrix} \text{Reactions} \end{bmatrix}$$

Based on BCs and applied displacements the final matrix becomes following.

$$K \times \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ u_4 \\ v_4 \\ 0 \\ v_5 \\ 0 \\ -0.01 \end{bmatrix} = \begin{bmatrix} 0 + R_1 \\ -375 + R_2 \\ 0 + R_3 \\ -1125 + R_4 \\ 0 + R_5 \\ -375 + R_6 \\ 0 \\ -1125 \\ 0 + R_7 \\ -1125 \\ 0 + R_8 \\ -375 + R_9 \end{bmatrix}$$

Wherever, we are restricting the displacement we will get the reactions at those nodes.

The above system can be reduced to 4 equations (u_4 v_4 v_5 -0.01) so after taking the corresponding rows,

$$1E10 \begin{bmatrix} 1.4583 & 0.3125 & -0.3125 & 0.1042 \\ 0.3125 & 1.4583 & -0.4167 & 0 \\ -0.3125 & -0.4167 & 1.4583 & -0.5208 \\ 0.1042 & 0 & -0.5208 & 0.5208 \end{bmatrix} \begin{bmatrix} u_4 \\ v_4 \\ v_5 \\ -0.01 \end{bmatrix} = \begin{bmatrix} 0 \\ -1125 \\ -1125 \\ -375 \end{bmatrix}$$

on solving we get.

$u_4 = 1.282\ E-4$

$v_4 = -1.13258E-3$

$v_5 = -3.86763\ E-3$

Reactions are,

$R_1 = -1.1797\ E6$

$R_2 = 2.6747\ E5$

$R_3 = -9.0812\ E6$

$R_4 = 1.1398\ E7$

$R_5 = 4.0288\ E6$

$R_6 = 2.0144\ E7$

$R_7 = 5.3418\ E5$

$R_8 = 5.698\ E6$

$R_9 = -3.18\ E7$

This is done using this –
after solving u we know all u's

so $\bar{R} = Ku - F$

$$\bar{R} = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \end{bmatrix}$$

Refer to MATLAB CODE for more details.