Homework 1: Basics of FE

Consider the following differential equation:

$$-u'' = f \text{ in } [0,1]$$

with the boundary conditions $u(0) = 0$ and $u(1) = \alpha$.

The Finite Element discretization is a 2-noded linear mesh given by the nodes $x_i = ih$ for $i = 0,1,\dots,n$ and $h = 1/n$.

1. Find the weak form of the problem. Describe the FE approximation u^h.
2. Describe the linear system of equations to be solved.
3. Compute the FE approximation u^h for $n = 3$, $f(x) = \sin x$ and $\alpha = 3$. Compare it with the exact solution $u(x) = \sin x + (3 - \sin 1)x$.

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1)

The strong form of the problem is:

$$A(u) = \left\{ -\frac{d^2u}{dx^2} = f(x) \right.$$

$$B(u) = \begin{cases} u - \bar{u} = 0 \ on\ x = 0,1 \\ \bar{u}(0) = 0, \bar{u}(1) = \alpha \end{cases}$$

u can be approximated as:

$$u \approx u^h = \sum_{i=1}^{N} N_i(x) u_i$$

Where N_i are linear piecewise shape functions:

$$N_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h} & x \in [x_{i-1}, x_i] \\ \dfrac{x_{i+1} - x}{h} & x \in [x_i, x_{i+1}] \\ 0 & otherwise \end{cases}$$

The derivatives of the shape functions are:

$$\frac{dN_i(x)}{dx} = \begin{cases} \dfrac{1}{h} & x \in [x_{i-1}, x_i] \\ \dfrac{-1}{h} & x \in [x_i, x_{i+1}] \\ 0 & otherwise \end{cases}$$

Applying the weighted residual method:

$$\int_0^1 W\left[-\frac{d^2u^h}{dx}\right]dx = \int_0^1 Wf dx$$

Integrating by parts:

$$\int_0^1 \frac{dW}{dx}\frac{du^h}{dx}\,dx = \left[W\frac{du^h}{dx}\right]_0^1 + \int_0^1 Wf dx$$

Substituting $u^h = \sum_{i=1}^{N} N_i(x)u_i$:

$$\int_0^1 \frac{dW}{dx}\left[\sum_{j=1}^{N}\frac{dN_j(x)}{dx}\,u_j\right]dx = \left[W\left[\sum_{j=1}^{N}\frac{dN_j(x)}{dx}\,u_j\right]\right]_0^1 + \int_0^1 Wf dx$$

Using the Galerkin method (W=N):

$$\sum_{i=1}^{N}\int_0^1 \frac{dN_i(x)}{dx}\left[\sum_{j=1}^{N}\frac{dN_j(x)}{dx}\,u_j\right]dx = \left[N_i(x)\left[\sum_{j=1}^{N}\frac{dN_j(x)}{dx}\,u_j\right]\right]_0^1 + \int_0^1 N_i(x)f dx$$

The first term in the right hand side of the equation is a "reaction" in the boundary nodes with a Dirichlet boundary condition. This term does not contribute to the solution ($u(0)$ and $u(1)$ are known) and can be computed after solving the system.

2)

For an arbitrary element "e":

$$\sum_{i=1}^{2}\int_{x_1^e}^{x_2^e}\frac{dN_i^e(x)}{dx}\left[\frac{dN_1^e(x)}{dx}\,u_1\right]dx = \int_{x_1^e}^{x_2^e} N_i^e(x)f dx$$

This system of equations can be expressed as:

$$\begin{bmatrix} K_{11}^e & K_{12}^e \\ K_{21}^e & K_{22}^e \end{bmatrix}\begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix} = \begin{bmatrix} f_1^e \\ f_2^e \end{bmatrix}$$

Where:

$$K_{11}^e = \int_{x_1^e}^{x_2^e}\frac{dN_1^e(x)}{dx}\frac{dN_1^e(x)}{dx}\,dx = \int_{x_1^e}^{x_2^e}\frac{1}{h^2}\,dx = \frac{1}{h}$$

$$K_{12}^e = K_{21}^e = \int_{x_1^e}^{x_2^e}\frac{dN_1^e(x)}{dx}\frac{dN_2^e(x)}{dx}\,dx = \int_{x_1^e}^{x_2^e}-\frac{1}{h^2}\,dx = -\frac{1}{h}$$

$$K_{22}^e = \int_{x_1^e}^{x_2^e}\frac{dN_2^e(x)}{dx}\frac{dN_2^e(x)}{dx}\,dx = \int_{x_1^e}^{x_2^e}\frac{1}{h^2}\,dx = \frac{1}{h}$$

$$f_1^e = \int_{x_1^e}^{x_2^e} N_1^e(x) \cdot \sin(x)\, dx = \int_{x_1^e}^{x_2^e} \frac{x_2^e - x}{h} \cdot \mathrm{f}(x)\, dx$$

$$f_2^e = \int_{x_1^e}^{x_2^e} N_2^e(x) \cdot \sin(x)\, dx = \int_{x_1^e}^{x_2^e} \frac{x - x_1^e}{h} \cdot \mathrm{f}(x)\, dx$$

The general system of equations is:

$$\begin{bmatrix} K_{11}^1 & K_{12}^1 & 0 & .. & .. & .. & .. \\ K_{21}^1 & K_{22}^1 + K_{11}^2 & K_{12}^2 & .. & .. & .. & .. \\ 0 & K_{21}^2 & K_{22}^2 + K_{11}^3 & .. & .. & .. & .. \\ : & : & : & & .. & .. & .. \\ : & : & : & : & K_{22}^{n-2} + K_{11}^{n-1} & K_{12}^{n-1} & 0 \\ : & : & : & : & K_{21}^{n-1} & K_{22}^{n-1} + K_{11}^{n} & K_{12}^{n} \\ : & : & : & : & 0 & K_{21}^{n} & K_{22}^{n} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ : \\ : \\ u_{n-1} \\ u_n \end{bmatrix} = \begin{bmatrix} f_1^1 + R_1 \\ f_2^1 + f_1^2 \\ f_2^2 + f_1^3 \\ : \\ : \\ f_2^{n-2} + f_1^{n-1} \\ f_2^{n-1} + R_n \end{bmatrix}$$

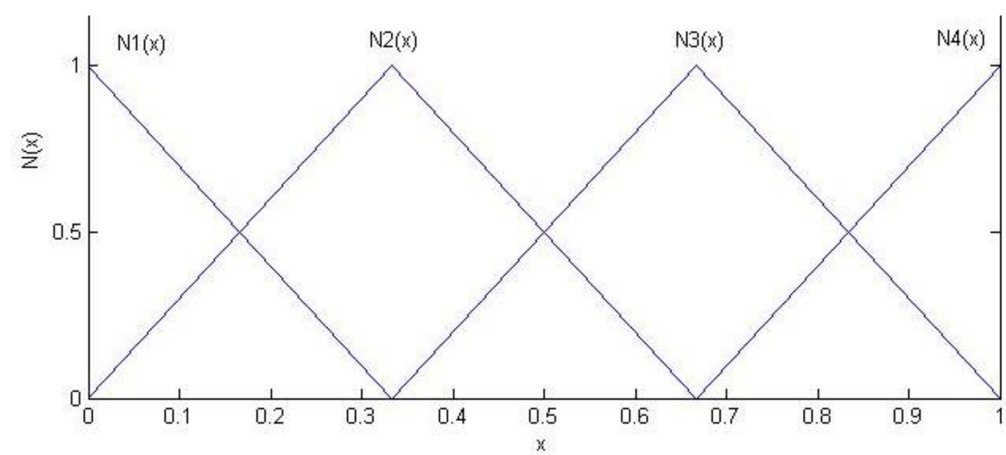
Where R_1 and R_n are the “reactions”. Since u_1 and u_n are already known, the linear system of equations that must be solved is:

$$\begin{bmatrix} K_{22}^1 + K_{11}^2 & K_{12}^2 & .. & .. & .. \\ K_{21}^2 & K_{22}^2 + K_{11}^3 & .. & .. & .. \\ : & : & & .. & .. \\ : & : & : & K_{22}^{n-2} + K_{11}^{n-1} & K_{12}^{n-1} \\ : & : & : & K_{21}^{n-1} & K_{22}^{n-1} + K_{11}^{n} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ : \\ : \\ u_{n-1} \end{bmatrix} = \begin{bmatrix} f_2^1 + f_1^2 - K_{21}^1 u_0 \\ f_2^2 + f_1^3 \\ : \\ : \\ f_2^{n-2} + f_1^{n-1} - K_{12}^n u_n \end{bmatrix}$$

3)

The mesh is composed by 3 linear 2-noded elements and 4 nodes.

The shape functions are:



For an arbitrary element:

$$\begin{cases} K_{11}^e = K_{22}^e = \dfrac{1}{h} = \dfrac{1}{0.33} = 3 \\ K_{12}^e = K_{21}^e = -\dfrac{1}{h} = -\dfrac{1}{0.33} = -3 \\ f_1^e = \displaystyle\int_{x_1^e}^{x_2^e} \frac{x_2^e - x}{h} \cdot \sin(x)\, dx = \cos(x_1^e) + \frac{\sin x_1^e - \sin x_2^e}{h} \\ f_2^e = \displaystyle\int_{x_1^e}^{x_2^e} \frac{x - x_1^e}{h} \cdot \sin(x)\, dx = -\cos(x_2^e) + \frac{\sin x_2^e - \sin x_1^e}{h} \end{cases}$$

The system of equations is:

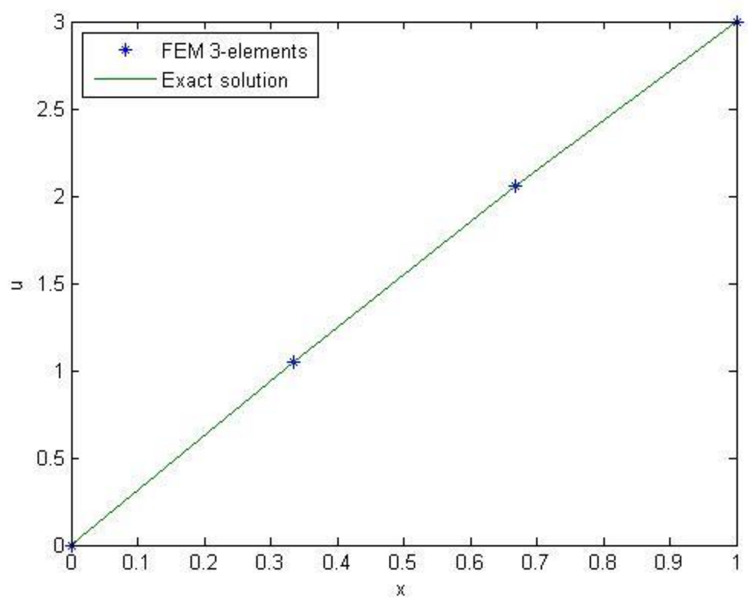
$$\begin{bmatrix} K_{22}^1 + K_{11}^2 & K_{12}^2 \\ K_{21}^2 & K_{22}^2 + K_{11}^3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_2^1 + f_1^2 - K_{21}^1 \cdot 0 \\ f_2^2 + f_1^3 - K_{12}^4 \alpha \end{bmatrix}$$

So:

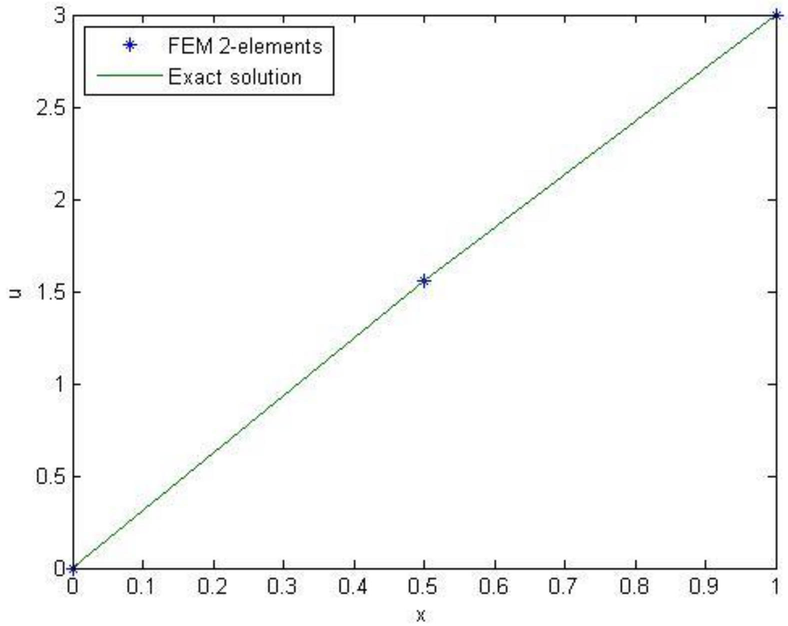
$$\begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix} = \begin{bmatrix} 0.0366 + 0.0714 \\ 0.0876 + 0.1166 + 3 * 3 \end{bmatrix}$$

The results are:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1.047 \\ 2.057 \end{bmatrix}$$



Since the exact solution of “u” is a line, we could have solved this problem just using 3 nodes and 2 elements:



The same result is obtained when using more elements: