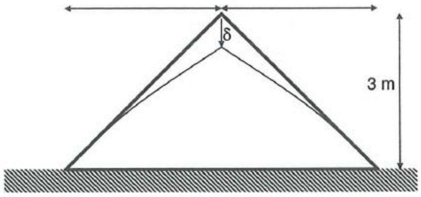

Figure 1: Geometry of the structure and prescribed displacement δ

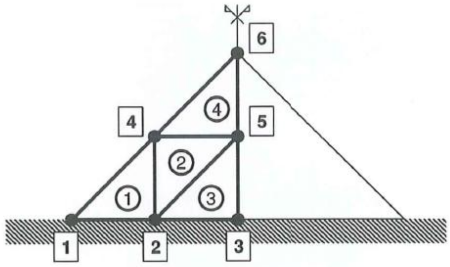

Figure 2: Mesh description. Node numbers are in squared boxes and element numbers in circles.

1. Strong form of the problem in the reduced domain. Indicate the Boundary Conditions in every edge

The strong form of this problem consists on the governing differential equations, the constitutive equation and the boundary conditions.

Differential equation:

$$\nabla\sigma + \rho b = 0$$

Constitutive equations, define the relationship between strain and stresses.

$$\sigma = D\varepsilon$$

The boundary conditions for this problem are restrictions or impositions of the displacements:

$$u = 0 \; for \; node \; 1\;,2,3,5 \; and \; 6$$

$$v = 0 \; for \; node \; 1{,}2 \; and \; 3$$

$$v = \delta \; for \; node \; 6, we \; have \; an \; imposed \; displacement$$

2. Arrays of the nodal coordinates X and the connectivitiy matrix T.

$$X = \begin{bmatrix} 0 & 0 \\ 1.5 & 0 \\ 3 & 0 \\ 1.5 & 1.5 \\ 3 & 1.5 \\ 3 & 3 \end{bmatrix} \qquad T = \begin{bmatrix} 2 & 4 & 1 \\ 4 & 2 & 5 \\ 3 & 5 & 2 \\ 5 & 6 & 4 \end{bmatrix}$$

3. Set up the linear system of equations corresponding to the discretitzation. How many degrees of freedom has the system to be solved?

This problem consist on a plane model, therefore the constitutive matrix has the structure:

$$D = \begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{21} & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}$$

The components of the matrix take the following form because the fact that its a plane stress problem and an isotropic material (same Young modulus and Poisson ratio en every direction).

$$d_{11} = d_{22} = \frac{E}{1-\nu^2}$$
$$d_{12} = d_{21} = \nu d_{11}$$
$$d_{33} = \frac{E}{2(1+\nu)} = G$$

The strains can be obtained from the derivatives of the displacements as following

$$\varepsilon_x = \frac{\partial u}{\partial x}$$
$$\varepsilon_y = \frac{\partial v}{\partial y}$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

For a FEM approximation using linear triangle elements the displacement field can be discretized as a function of the 3 nodal displacements of the element and the shape function:

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$
$$v = N_1 v_1 + N_2 v_2 + N_3 v_3$$

Each node defines a linear displacement field that can be written for each element as:

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y$$
$$v = \alpha_4 + \alpha_5 x + \alpha_3 y$$

Solving the system we get a discretized function for the displacement for each element:

$$u_h = \frac{1}{2A^{(e)}}[(a_1 + b_1 x + c_1 y)u_1 + (a_2 + b_2 x + c_2 y)u_2 + (a_3 + b_3 x + c_3 y)u_3]$$

$$a_i = x_j y_k - x_k y_j$$
$$b_i = y_j - y_k$$
$$c_i = x_k - x_j$$

Then, the shape function takes the following form

$$N_i = \frac{1}{2A^{(e)}}(a_i + b_i x + c_i y)$$

We can calculate the discretized strains

$$\varepsilon_x = \frac{\partial u}{\partial x}$$
$$\varepsilon_y = \frac{\partial v}{\partial y}$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Collecting the derivative terms from the shape function into the matrix B

$$B^{(e)} = \frac{1}{2A^{(e)}} \begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix}$$

The stresses can be related to the strains using the constitutive matrix, therefore:

$$\sigma = D\varepsilon = DBa^{(e)}$$

Applying the virtual work principle we can find an equation that describes the nodal equilibrium for the elements (r=thickness, t=traction, b= body force).

$$\iint_A^{(e)} B^T \sigma r dA - \iint_A^{(e)} N^T b r dA - \oint_l N^T r t ds = q^{(e)}$$

The nodal forces are the result of the integration of the element deformation (first integral), the body forces (second integral) and the surface tractions (third integral). Substituting the stress in terms of nodal displacements, and assuming no initial stresses, strains or surface tractions (as in the problem) we can collect the terms in matrices to form the following linear system.

$$K^{(e)} a^{(e)} - f^{(e)} = q^{(e)}$$
$$K^{(e)} = \iint_{A^{(e)}} B^T.D.B.r.dA$$
$$f^{(e)} = N^T.b.r.dA$$

We have to compute the stiffness matrix for every element and do the assembly to get the global stiffness matrix. To be able to solve the linear system we also need to impose the boundary conditions in every node. As we have a linear elastic problem every node has two degrees of freedom one for vertical displacements and the other for horizontal displacements.

The global stiffness matrix looks like this:

Global	1	2	3	4	5	6
1	K_{33}^1	K_{31}^1		K_{32}^1		
2	K_{13}^1	$K_{11}^1+K_{22}^2+K_{33}^3$	K_{32}^3	$K_{12}^1+K_{21}^2$	$K_{23}^2+K_{32}^3$	
3		K_{13}^3	K_{11}^3		K_{12}^3	
4	K_{23}^1	$K_{12}^1+K_{12}^2$		$K_{22}^1+K_{11}^2+K_{33}^4$	$K_{13}^2+K_{31}^4$	K_{32}^4
5		$K_{23}^2+K_{23}^3$	K_{21}^3	$K_{31}^2+K_{13}^4$	$K_{33}^2+K_{22}^3+K_{11}^4$	K_{12}^4
6				K_{23}^4	K_{21}^4	K_{23}^4

As we have seen every stiffness matrix is a 2x2 matrix, so in reality we have a 12x12 global stiffness matrix.

Global	
1	$u_1=0$ $v_1=0$
2	$u_2=0$ $v_2=0$
3	$u_3=0$ $v_3=0$
4	u_4 v_4
5	$u_5=0$ v_5
6	$u_6=0$ $v_6=-\delta$

When applying the boundary conditions the only unknown displacements are the horizontal displacements in node 4 and vertical displacements in nodes 4 and 5. The system will be reduced to three equations. Knowing that the only prescribed displacement different to 0 corresponds to the vertical displacement in node 6 the reduced system that we have to take into account subtracting that value to the global force vector.

4. Compute the FE approximation u^h. Use $E=10GPa, \nu=0.2, \delta=10^{-2}m$ and $\rho g=10^3 N/m^2$

As we have seen the linear system reduces to a matrix 3x3 after imposing the boundary conditions, so we don't need to compute the stiffness matrix for all the elements. Here we can see the matrixes that we need to solve.

$$K_{11}^1=0.22\begin{bmatrix}3.28\cdot10^{10} & -1.4\cdot10^{10}\\ -1.4\cdot10^{10} & 3.28\cdot10^{10}\end{bmatrix}=K_{11}^2=K_{11}^4$$

$$K_{11}^2=0.22\begin{bmatrix}3.28\cdot10^{10} & -1.4\cdot10^{10}\\ -1.4\cdot10^{10} & 3.28\cdot10^{10}\end{bmatrix}$$

$$K_{13}^2=0.22\begin{bmatrix}-2.34\cdot10^{10} & 9.38\cdot10^{10}\\ 4.68\cdot10^{9} & -9.382\cdot10^{9}\end{bmatrix}=K_{13}^4$$

$$K_{31}^2=0.22\begin{bmatrix}-2.34\cdot10^{10} & 4.68\cdot10^{9}\\ 9.38\cdot10^{9} & -9.38\cdot10^{9}\end{bmatrix}=K_{31}^4$$

$$K_{33}^{2} = 0.22\begin{bmatrix} 2.34\cdot 10^{10} & 0 \\ 0 & 9.38\cdot 10^{9} \end{bmatrix} = K_{33}^{4}$$

$$K_{22}^{3} = 0.22\begin{bmatrix} 9.38\cdot 10^{9} & 0 \\ 0 & 2.34\cdot 10^{9} \end{bmatrix} = K_{22}^{1}$$

$$K_{32}^{4} = 0.22\begin{bmatrix} 0 & -4.68\cdot 10^{9} \\ -9.38\cdot 10^{9} & 0 \end{bmatrix}$$

$$0.22\begin{bmatrix} 6.558\cdot 10^{10} & -1.4\cdot 10^{10} & 1.4\cdot 10^{10} \\ -1.4\cdot 10^{10} & 6.558\cdot 10^{10} & -1.89\cdot 10^{10} \\ 1.4\cdot 10^{10} & -1.89\cdot 10^{10} & 6.558\cdot 10^{10} \end{bmatrix}$$

The body forces for each node are:

$$f_i = \frac{(At)^e}{3}\begin{bmatrix} 0 \\ -\rho g \end{bmatrix}$$

All the elements have the same area the force vector is the same for each one of them

$$f^e = \begin{bmatrix} 0 \\ -375 \\ 0 \\ -375 \\ 0 \\ -375 \end{bmatrix}$$

Doing the assembly of the global vector using the connectivity matrix we get;

$$f = \begin{bmatrix} 0 \\ -375 \\ 0 \\ -1125 \\ 0 \\ -375 \\ 0 \\ -1125 \\ 0 \\ -1125 \\ 0 \\ -375 \end{bmatrix}$$

The reduce system that we have to solve is:

$$0.22\begin{bmatrix} 6.558\cdot 10^{10} & -1.4\cdot 10^{10} & 1.4\cdot 10^{10} \\ -1.4\cdot 10^{10} & 6.558\cdot 10^{10} & -1.89\cdot 10^{10} \\ 1.4\cdot 10^{10} & -1.89\cdot 10^{10} & 6.558\cdot 10^{10} \end{bmatrix}\begin{bmatrix} u_4 \\ v_4 \\ v_5 \end{bmatrix} = 10^{7}\begin{bmatrix} -1.0417 \\ -0.0001 \\ -5.2084 \end{bmatrix}$$

Solving the system we can get the unknown horizontal and vertical displacements, getting as a solution:

$$\begin{Bmatrix} u_4 \\ v_4 \\ v_5 \end{Bmatrix} = \begin{bmatrix} -0.0001 \\ -0.0011 \\ -0.0039 \end{bmatrix}$$