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Finite Elements Homework #1

Consider the following differential equation:

-u''=f in] 0, 1[

With the boundary conditions $u(0)=0$ and $u(1)=\alpha$.

The Finite Element discretization is a 2-noded linear mesh given by the nodes $x_i=ih$ for i=0, 1, ..., n and h=1/n.

1. Find the weak form of the problem. Describe the FE approximation u^h.

As we have a Poisson 1D equation with the form:

$$\frac{du^2}{dx^2}+f=0$$

Multiplying by a test function $w(x)$ and including in the equation the Dirichlet boundary conditions we obtain the weak form:

$$\int_{\Omega} \omega(x)\cdot\left[\frac{du^2}{dx^2}+f\right]d\Omega+\int_{\Gamma} \tilde{\omega}(x)\frac{du}{dx}d\Gamma=0$$

The u^h approximation is describe as follows:

$$u(x) = u^h(x)=\sum_{j=1}^{n} u_j N(x)_j$$

2. Describe the linear system of equations to be solved.

$$\int_{\Omega} \omega(x)\cdot\left[\frac{du^2}{dx^2}+f\right]d\Omega+\int_{\Gamma} \tilde{\omega}(x)\frac{du}{dx}d\Gamma=0$$

First we substitute $u(x) = u^h(x)$ and we get

$$\int_{\Omega} \omega(x)\cdot\left[\frac{du^{h^2}}{dx^2}+f\right]d\Omega+\int_{\Gamma} \tilde{\omega}(x)\frac{du^h}{dx}d\Gamma=0$$

And then we substitute $u^h(x)$

$$\int_{\Omega} N_i\cdot\left[\frac{d\,[\sum_{j=1}^{n} u_j N(x)_j]^2}{dx^2}+f\right]d\Omega+\int_{\Gamma} N_i\frac{d(\sum_{j=1}^{n} u_j N(x)_j)}{dx}d\Gamma=0$$

Integrating by parts we get:

$$-\int_{\Omega} N_i\cdot\left[\frac{d\,[\sum_{j=1}^{n} u_j N(x)_j]^2}{dx^2}\right]d\Omega+\int_{\Omega} N_i\cdot f\,d\Omega+\int_{\Gamma} N_i\frac{d(\sum_{j=1}^{n} u_j N(x)_j)}{dx}d\Gamma=0$$

Where the lineal system to solve reads:

$$K_{ij} = \int_{\Omega} N_i \cdot \left[\frac{d\,[\sum_{j=1}^{n} N(x)_j]^2}{dx^2}\right] d\Omega$$

$$f_i = \int_{\Omega} N_i \frac{d(\sum_{j=1}^{n} N(x)_j)}{dx} f\,d\Omega$$

3. Compute the FE approximation u^h for n=3, *f(x) = sinx* and *α=3.* Compare with the exact solution*, u(x) = sinx + (3-sin1)x.*

   We will have three elements (n=3) and 4 nodes.

$$u^h(x) = \sum_{j=1}^{n} u_j N(x)_j$$

   We define *N(x)* as:

$$N_1 = \frac{x_2 - x}{l} \qquad \frac{dN_1}{dx} = -\frac{1}{l}$$

$$N_2 = \frac{x - x_1}{l} \qquad \frac{dN_2}{dx} = \frac{1}{l}$$

   Solving for each element:

$$K_{ij} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$f_i^1 = \begin{bmatrix} \int_0^{1/3} \frac{x_2 - x}{l} \sin(x) \\ \int_0^{1/3} \frac{x - x_1}{l} \sin(x) \end{bmatrix} = \begin{bmatrix} 0.01842 \\ 0.03663 \end{bmatrix}$$

$$f_i^2 = \begin{bmatrix} \int_{1/3}^{2/3} \frac{x_2 - x}{l} \sin(x) \\ \int_{1/3}^{2/3} \frac{x - x_1}{l} \sin(x) \end{bmatrix} = \begin{bmatrix} 0.07143 \\ -0.08764 \end{bmatrix}$$

$$f_i^3 = \begin{bmatrix} \int_{2/3}^{1} \frac{x_2 - x}{l} \sin(x) \\ \int_{2/3}^{1} \frac{x - x_1}{l} \sin(x) \end{bmatrix} = \begin{bmatrix} 0.11658 \\ 0.12900 \end{bmatrix}$$

   So ensambling the local solutions, we get the global system to be solved:

$$3\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ u_2 \\ u_3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0.01842 \\ 0.03663 + 0.07143 \\ -0.08764 + 0.11658 \\ 0.12900 \end{bmatrix}$$

   Solving:

$$\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0.57686 \\ 1.04567 \end{bmatrix}$$

Comparing with the exact solution:

$$u_3 = \sin(x) + (3 - sin1)x \,|_{\,x=1} = 1.0467$$