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Dear teachers,

Due to the insufficient function of my computer, I just saved the key step for getting the answer. I would be very happy to hear from your suggestions about my work.

Thanks for your attention.

Best,

Yuyang

1 Find the weak form of the problem. Describe the FE approximation u^h.

$$\int_0^1 w(\frac{d^2u}{dx^2} + f)dx = 0$$

And after proper operation, I get the weak form:

$$\int_0^1 \frac{du}{dx}\frac{dw}{dx}dx = \left[w\frac{du}{dx}\right]_1 - \left[w\frac{du}{dx}\right]_0 + \int_0^1 wf\,dx$$

FE approximation:

$$u \cong u^h = \sum_{j=0}^{n} N_j u_j = N_0 u_0 + N_1 u_1 + N_2 u_2 + \cdots + N_n u_n$$

2 Describe the linear system of equations to be solved.

By Galerkin method and discretization process,the weak form can be written as:

$$\sum_{j=0}^{n} \int_0^1 \frac{dN_i}{dx}\frac{dN_j}{dx}dx u_j = \left[N_i\frac{du}{dx}\right]_1 - \left[N_i\frac{du}{dx}\right]_0 + \int_0^1 N_i f\,dx$$

$$\left[\frac{du}{dx}\right]_1 = -\bar{q}$$

$$\left[\frac{du}{dx}\right]_0 = -q_0$$

For this part, take it as a general case in the 2-noded element, we can get the stiffness matrix K(e) and the equivalent nodal flux vector f(e).

$$K^e = \begin{bmatrix} K_{11}^e & K_{12}^e \\ K_{21}^e & K_{22}^e \end{bmatrix}$$

$$f^e = \begin{bmatrix} f_1^e \\ f_2^e \end{bmatrix}$$

$$K_{ij}^e = (-1)^{i+j}\left(\frac{k}{l}\right)^e$$

$$f_e^i = \int_{x1}^{x2} N_i^e Q\,dx$$

We obtain the global stiffness matrix K and the global equivalent nodal flux vector.

$$\frac{1}{h} * \begin{bmatrix} 1 & -1 & \dots & 0 & 0 \\ -1 & 2 & & 0 & 0 \\ \vdots & & \ddots & \vdots & \\ 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-1} \\ u_n \end{Bmatrix} = \begin{Bmatrix} f_1^1 + q_0 \\ f_2^1 + f_1^2 \\ \vdots \\ f_2^{n-2} + f_1^{n-1} \\ f_2^n - \bar{q} \end{Bmatrix}$$

3 Compute the FE approximation u^h for n=3, f(x)=sinx and α=3. compare it with the exact solution, u(x)=sinx+(3-sin1)x.

Global	local	domain	$\frac{dN^e}{dx}$
N0	N_1^1	[0,1/3]	-3
N0	0	[1/3,1]	0
N1	N_2^1	[0,1/3]	3
N1	N_1^2	[1/3,2/3]	-3
N2	N_2^2	[1/3,2/3]	3
N2	N_1^3	[2/31]	-3
N3	N_2^3	[2/3,1]	3
N3	0	[0,2/3]	0

According to the linear system of the equations and after some operations, we can get the final linear equations as follows:

$$3 * \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{Bmatrix} u_0 \\ u_1 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{pmatrix} \int_0^{\frac{1}{3}} (1-3x)\sin(x)\,dx + q_0 \\ \int_0^{\frac{1}{3}} (3x)\sin(x)\,dx + \int_{1/3}^{\frac{2}{3}} (2-3x)\sin(x)\,dx \\ \int_{1/3}^{2/3} (3x-1)\sin(x)\,dx + \int_{2/3}^{1} (-3x+3)\sin(x)\,dx \\ \int_{2/3}^{1} (3x-2)\sin(x)\,dx - \bar{q} \end{pmatrix}$$

By computing these equations, we get that $\begin{Bmatrix} q_0 \\ \bar{q} \\ u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} -3.1585 \\ -2.6989 \\ 1.0467 \\ 2.0574 \end{Bmatrix}$

On the other hand, the exact solution for u1 and u2 is 1.0467 and 2.0574.