Homework-1, Finite Element Methods

Carles Duñó

Considering the following differential equation:

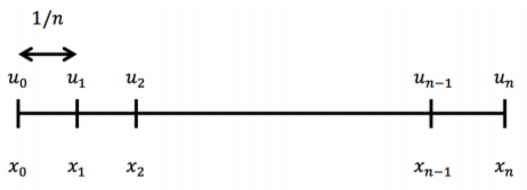
$$-u'' = f \qquad \rightarrow \qquad -\frac{\partial^2 u}{\partial x^2} = f \qquad \text{in } [0,1]$$

With the boundary conditions:

$$u(0) = 0 \text{ and } u(1) = \alpha$$

The finite element discretization is a 2-noded linear mesh given by the nodes:

$$x_i = ih, \text{ and } i=0,1,\ldots,n \qquad h = 1/n$$



Q 1: Find the weak form of the problem. Describe the FE approximation u^h.

Multiplying both sides by some weighting function w and taking integral on both sides over the domain Ω [0,1]:

$$\int_0^1 W_i \frac{\partial^2 u}{\partial x^2} dx = \int_0^1 W_i f dx$$

Integrating by parts,

$$\int_0^1 \left\{ \left[\frac{\partial^2 u}{\partial x^2} \right] \cdot W \right\} dx = W \frac{\partial u}{\partial x}\bigg|_0^1 + \int_0^1 \frac{\partial W}{\partial x} \frac{\partial u}{\partial x} dx$$

The weak form of the equation is:

$$\int_0^1 \frac{\partial W}{\partial x} \frac{\partial u}{\partial x} dx = \int_0^1 W \cdot f \, dx + W \frac{\partial u}{\partial x}\bigg|_0^1$$

Q 2: Describe the linear system of equations to be solved.

We can use a piecewise linear approximations for u in the following way:

$$u \approx u^h = \sum_{i=0}^{1} N_j(x) \cdot a_j$$

$$W(x) = N_i(x)$$

Shape functions can be generalized as:

$$N_1{}^{(e)}(x) = \frac{x_2{}^{(e)} - x}{l^{(e)}} \qquad ; \qquad N_2{}^{(e)}(x) = \frac{x - x_1{}^{(e)}}{l^{(e)}}$$

The global set equation

$$\int_0^1 \frac{\partial W_i}{\partial x} \sum_{j=0}^{n} \frac{\partial N_j}{\partial x} a_j \, dx = \int_0^1 W_i \cdot f dx + \left[W_i \frac{\partial u}{\partial x}\right]_1 - \left[W_i \frac{\partial u}{\partial x}\right]_0$$

$$\int_0^1 \frac{\partial N_i}{\partial x} \left(\frac{\partial N_1}{\partial x} a_1 + \frac{\partial N_2}{\partial x} a_2 + \cdots + \frac{\partial N_n}{\partial x} a_n\right) dx = \int_0^1 N_i \cdot f dx + \left[N_i \frac{\partial u}{\partial x}\right]_1 - \left[N_i \frac{\partial u}{\partial x}\right]_0$$

Solving for a 2-noded linear element:

$$k_{ij}^{(e)} = \int_0^{1/3} \frac{\partial N_i{}^{(e)}}{\partial x} \left(\frac{\partial N_1{}^{(e)}}{\partial x} a_1 + \frac{\partial N_2{}^{(e)}}{\partial x} a_2\right) dx$$

$$f_i^{(e)} = \int_0^{\frac{1}{3}} N_i{}^{(e)} \cdot f dx + \left[N_i \frac{\partial u}{\partial x}\right]_1 - \left[N_i \frac{\partial u}{\partial x}\right]_0$$

Linear System of Equation can be defined in following way:

$$\begin{bmatrix} k_{11} & \cdots & k_{1n} \\ \vdots & \ddots & \vdots \\ k_{n1} & \cdots & k_{nn} \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$$

$$[\boldsymbol{k}] \cdot [\boldsymbol{a}] = [\boldsymbol{f}]$$

Q 3: Solve the system for n=3

Solving for n=3 we get the following discretization:

$$\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} + k_{11}^{(2)} & k_{12}^{(2)} & 0 \\ 0 & k_{21}^{(2)} & k_{22}^{(2)} + k_{11}^{3} & k_{12}^{3} \\ 0 & 0 & k_{21}^{3} & k_{22}^{3} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} f_1^{(1)} - \bar{q}_0 \\ f_2^{(1)} + f_1^{(2)} \\ f_2^2 + f_1^3 \\ f_2^{(3)} + \bar{q}_1 \end{bmatrix}$$

$$\frac{\partial N_1{}^{(1)}}{\partial x} = \frac{-1}{h^{(1)}} \qquad \frac{\partial N_1{}^{(2)}}{\partial x} = \frac{-1}{h^{(2)}} \qquad \frac{\partial N_1{}^{(3)}}{\partial x} = \frac{-1}{h^{(3)}}$$

$$\frac{\partial N_2{}^{(1)}}{\partial x} = \frac{1}{h^{(1)}} \qquad \frac{\partial N_2{}^{(2)}}{\partial x} = \frac{1}{h^{(2)}} \qquad \frac{\partial N_2{}^{(3)}}{\partial x} = \frac{1}{h^{(3)}}$$

$$k_{11} = \int_0^{1/3} \frac{\partial N_1{}^{(1)}}{\partial x} \frac{\partial N_1{}^{(1)}}{\partial x} dx = \frac{1}{h^{2(1)}} \cdot \frac{1}{3} = \frac{1}{3h^{2(1)}} = 3$$

$$k_{12} = \int_0^{1/3} \frac{\partial N_1{}^{(1)}}{\partial x} \frac{\partial N_2{}^{(1)}}{\partial x} dx = \frac{1}{3h^{3(2)}} = -3$$

$$k_{22} = \int_0^{1/3} \frac{\partial N_2{}^{(1)}}{\partial x} \frac{\partial N_2{}^{(1)}}{\partial x} dx = \frac{1}{3h^{2(2)}} = \frac{1}{3(1/3)^2} = 3$$

$$k_{11} = k_{22}$$

$$k_{12} = k_{21}$$

For i=1

$$\int_0^{1/3} \frac{\partial N_1{}^{(1)}}{\partial x} \left(\frac{\partial N_1{}^{(1)}}{\partial x} a_1 + \frac{\partial N_2{}^{(1)}}{\partial x} a_2 \right) dx = \int_0^{1/3} N_1{}^{(1)} \cdot f dx + R_0$$

For i=2

$$\int_0^{1/3} \frac{\partial N_2{}^{(1)}}{\partial x} \left(\frac{\partial N_1{}^{(1)}}{\partial x} a_1 + \frac{\partial N_2{}^{(1)}}{\partial x} a_2 \right) dx + \int_{1/3}^{2/3} \frac{\partial N_1{}^{(2)}}{\partial x} \left(\frac{\partial N_1{}^{(2)}}{\partial x} a_1 + \frac{\partial N_2{}^{(2)}}{\partial x} a_2 \right) dx = \int_{1/3}^{2/3} N_1{}^{(1)} \cdot f dx + \int_0^{1/3} N_1{}^{(2)} \cdot f dx$$

For i=3

$$\int_{1/3}^{2/3} \frac{\partial N_2{}^{(2)}}{\partial x}\left(\frac{\partial N_1{}^{(2)}}{\partial x} a_1 + \frac{\partial N_2{}^{(2)}}{\partial x} a_2\right) dx + \int_{1/3}^{2/3} \frac{\partial N_1{}^{(3)}}{\partial x}\left(\frac{\partial N_1{}^{(3)}}{\partial x} a_1 + \frac{\partial N_2{}^{(3)}}{\partial x} a_2\right) dx = \int_{1/3}^{2/3} N_1{}^{(1)} \cdot f dx + \int_{0}^{1/3} N_1{}^{(2)} \cdot f dx$$

For i=4

$$\int_{2/3}^{1} \frac{\partial N_2{}^{(3)}}{\partial x}\left(\frac{\partial N_1{}^{(3)}}{\partial x} a_1 + \frac{\partial N_2{}^{(3)}}{\partial x} a_2\right) dx = \int_{2/3}^{1} N_2{}^{(3)} \cdot f dx + R_1$$

Computing [f]:

$$f(x) = sin(x);\ \alpha = 3$$

$$f_1{}^{(1)} = \int_0^{1/3} N_1{}^{(1)} \cdot f(x)\, dx = \int_0^{1/3} \frac{x_2{}^{(1)} - x}{h^{(1)}} \cdot sin(x)\, dx$$

$$= -\frac{x_2{}^{(1)}}{h^{(1)}} \cdot cos(x)\Big|_0^{1/3} - \frac{1}{h^{(1)}} \cdot (\sin(x) - x \cdot cos(x)\Big|_0^{1/3} = -1 - 3 \cdot \sin(1/3)$$

$$= 0.0184$$

$$f_2{}^{(1)} = \int_0^{1/3} N_2{}^{(1)} \cdot f(x)\, dx = \int_0^{1/3} \frac{x - x_1{}^{(1)}}{h^{(1)}} \cdot sin(x)\, dx = 3\left(\sin(1/3) - 1/3 \cdot cos(1/3)\right)$$

$$= 3 \cdot \sin(1/3) - cos(1/3) = 0.0366$$

$$f_1{}^{(2)} = \int_{\frac{1}{3}}^{2/3} N_1{}^{(2)} \cdot f(x)\, dx = \int_{\frac{1}{3}}^{2/3} \frac{x_3 - x}{h} \cdot sin(x)\, dx = -\frac{x_3}{h} \cdot cos(x)\Big|_{\frac{1}{3}}^{2/3} - \frac{1}{h} \cdot (\sin(x) -$$

$$x \cdot cos(x)\Big|_{\frac{1}{3}}^{\frac{2}{3}} = 0.0714$$

$$f_2{}^{(2)} = \int_{\frac{1}{3}}^{2/3} N_1{}^{(2)} \cdot f(x)\, dx = \int_{\frac{1}{3}}^{2/3} \frac{-x_2}{h} \cdot sin(x)\, dx + \int_{\frac{1}{3}}^{2/3} \frac{x}{h} \cdot sin(x)\, dx$$

$$= -\frac{x_2}{h} \cdot cos(x)\Big|_{\frac{1}{3}}^{2/3} + \frac{1}{h} \cdot (\sin(x) - x \cdot cos(x)\Big|_{\frac{1}{3}}^{\frac{2}{3}} = cos\left(\frac{2}{3}\right) + 3sin\left(\frac{2}{3}\right) - 3sin\left(\frac{1}{3}\right)$$

$$= 0.0877$$

$$f_1^{(3)} = \int_{\frac{2}{3}}^{1} \frac{x_4}{h} \cdot sin(x)\, dx + \int_{\frac{2}{3}}^{1} \frac{-x}{h} \cdot sin(x)\, dx$$

$$= -\frac{x_4}{h} \cdot cos(x)\Big|_{\frac{2}{3}}^{1} - \frac{1}{h} \cdot (\sin(x) - x \cdot cos(x)\Big|_{\frac{2}{3}}^{1}$$

$$= \cos\left(\frac{2}{3}\right) - 3\sin(1) + 3\sin\left(\frac{2}{3}\right) = 0.1163$$

$$f_2^{(3)} = \int_{\frac{2}{3}}^{1} \frac{-x_3}{h} \cdot sin(x)\, dx + \int_{\frac{2}{3}}^{1} \frac{x}{h} \cdot sin(x)\, dx = \frac{x_3}{h} \cdot cos(x)\Big|_{\frac{2}{3}}^{1} + \frac{1}{h} \cdot (\sin(x) - xcos(x)\Big|_{\frac{2}{3}}^{1} =$$

$$= -\cos(1) + 3\sin(1) - 3sin\left(\frac{2}{3}\right) = 0.1291$$

The matrix reads

$$3 \cdot \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \mathrm{a}_2 \\ a_3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0.0184 + R_0 \\ 0.108 \\ 0.2042 \\ 0.1291 + R_1 \end{bmatrix}$$

$$a_2 = 1.0466$$

$$a_3 = 2.0573$$

With the exact solution:

$$u(x) = \sin(x) + (3 - \sin(1)) \cdot x$$

$$u(0) = 0$$

$$u(^1/_3) = 1.0467$$

$$u(^2/_3) = 2.0573$$

$$u(1) = 3$$

We get the same results for all values of u from analytical solution.