Finite Elements

Homework 2: "Plane Elasticity"

1. Describe the strong form of the problem in the reduced domain (left half). Indicate accurately the Boundary Conditions in every edge.

The strong form of this problem consists in three parts:

- Differential equation for a steady state problem.

$$\boldsymbol{div(\sigma) + \rho \cdot b = 0}$$

- Constitutive equation to relate strains with strains.

$$\boldsymbol{\sigma} = \mathbf{D}\ \boldsymbol{\varepsilon}$$

Where:

$$\mathbf{D} = \begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{21} & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}$$

And for an isotropic elasticity in plane stress:

$$d_{11} = d_{22} = \frac{E}{1-\nu^2}$$
$$d_{12} = d_{21} = \nu d_{11}$$
$$d_{33} = \frac{E}{2(1+\nu)} = G$$

And the strains can be computed as the derivatives of the displacements

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$
$$\varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xz} = \gamma_{yz} = 0$$

- Boundary Conditions
  - Bottom: $U_x = U_y = 0$ Nodes: 1,2, and 3
  - Symmetry: $U_x = 0$ Nodes: (3), 5, and 6
  - Top: $U_y = \delta$ Node: 6

2. Describe the mesh shown in figure 2 by giving the arrays of nodal coordinates X and the connectivity matrix T. In order to simplify the computations select the local numbering of nodes such that, in every element, the node in the right angle vertex has local number equal to 1.

Nodal coordinates

$$X=\begin{pmatrix} 0 & 0 \\ 1.5 & 0 \\ 3 & 0 \\ 1.5 & 1.5 \\ 3 & 1.5 \\ 3 & 3 \end{pmatrix}$$

Row: element, Col: coordinates x, y

Connectivity Matrix

$$T=\begin{pmatrix} 2 & 4 & 1 \\ 4 & 2 & 5 \\ 3 & 5 & 2 \\ 5 & 6 & 4 \end{pmatrix}$$

Row: element, Col: nodes

3. Set up the linear system of equations corresponding to the discretization in figure 2. How many degrees of freedom has the system to be solved

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$
$$v = N_1 v_1 + N_2 v_2 + N_3 v_3$$

The FE approximation can be written in a linear form as

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y$$
$$v = \alpha_4 + \alpha_5 x + \alpha_6 y$$

From the approximation for u^h we can obtain

$$u_1 = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1$$
$$u_2 = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2$$
$$u_3 = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3$$

Solving for α_1, α_2 and α_3 and substituting gives

$$u = \frac{1}{2A^{(e)}}\left[(a_1 + b_1 x + c_1 y)u_1 + (a_2 + b_2 x + c_2 y)u_2 + (a_3 + b_3 x + c_3 y)u_3\right]$$

Where $A^{(e)}$ is the element area and

$$a_i = x_j y_k - x_k y_j \quad , \quad b_i = y_j - y_k \quad , \quad c_i = x_k - x_j \quad ; \quad i,j,k = 1,2,3$$

From the linear approximation it can be deduced

$$\boxed{N_i = \frac{1}{2A^{(e)}}(a_i + b_i x + c_i y)} \quad , \quad i = 1,2,3$$

The discretization of the strain field can be computed as

$$\begin{aligned}
\varepsilon_x &= \frac{\partial u}{\partial x} = \frac{\partial N_1}{\partial x}u_1 + \frac{\partial N_2}{\partial x}u_2 + \frac{\partial N_3}{\partial x}u_3 \\
\varepsilon_y &= \frac{\partial v}{\partial y} = \frac{\partial N_1}{\partial y}v_1 + \frac{\partial N_2}{\partial y}v_2 + \frac{\partial N_3}{\partial y}v_3 \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial N_1}{\partial y}u_1 + \frac{\partial N_1}{\partial x}v_1 + \frac{\partial N_2}{\partial y}u_2 + \frac{\partial N_2}{\partial x}v_2 + \frac{\partial N_3}{\partial y}u_3 + \frac{\partial N_3}{\partial x}v_3
\end{aligned}$$

Collecting the derivative terms of the shape function in to the matrix B

$$\mathbf{B} = \frac{1}{2A^{(e)}} \begin{bmatrix} b_1 & 0 & \vdots & b_2 & 0 & \vdots & b_3 & 0 \\ 0 & c_1 & \vdots & 0 & c_2 & \vdots & 0 & c_3 \\ c_1 & b_1 & \vdots & c_2 & b_2 & \vdots & c_3 & b_3 \end{bmatrix}$$

Discretization of the stress field

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} = \mathbf{DBa}^{(e)}$$

With

$$\mathbf{a}^{(e)} = \begin{Bmatrix} \mathbf{a}_1^{(e)} \\ \mathbf{a}_2^{(e)} \\ \mathbf{a}_3^{(e)} \end{Bmatrix} \qquad \text{with} \qquad \mathbf{a}_i^{(e)} = \begin{Bmatrix} u_i \\ v_i \end{Bmatrix}$$

Applying the virtual work principle we find an equation which equilibrate the nodal forces (r=thickness, t=traction, b= body force).

$$\iint_{A^{(e)}} \mathbf{B}^T \boldsymbol{\sigma} t \, dA - \iint_{A^{(e)}} \mathbf{N}^T \mathbf{b} t \, dA - \oint_{l^{(e)}} \mathbf{N}^T \mathbf{t} t \, ds = \mathbf{q}^{(e)}$$

Substituting the stress in terms of nodal displacements, and assuming no initial stresses, strains or surface tractions, it can be formed the following linear system of equations.

$$\mathbf{K}^{(e)}\mathbf{a}^{(e)} - \mathbf{f}^{(e)} = \mathbf{q}^{(e)}$$

$$\boxed{\mathbf{K}^{(e)} = \iint_{A^{(e)}} \mathbf{B}^T\mathbf{D}\ \mathbf{B}t\, dA}$$

$$\mathbf{f}_{b_i}^{(e)} = \iint_{A^{(e)}} \mathbf{N}_i^T \mathbf{b}\ t\, dA$$

*The system has 6 nodes with 2 directions. And when applying the BC the only unknowns leftare the vertical displacement in nodes 4 & 5 and the horitzontal one in node 4, with that we can reduce the system to a **3 equations**.*

$$\begin{bmatrix} K_{33}^1 + K_{11}^2 + K_{55}^4 & K_{34}^1 + K_{12}^2 + K_{56}^4 & K_{16}^2 + K_{52}^4 \\ K_{43}^1 + K_{21}^2 + K_{65}^4 & K_{44}^1 + K_{22}^2 + K_{66}^4 & K_{26}^2 + K_{62}^4 \\ K_{61}^2 + K_{25}^4 & K_{62}^2 + K_{26}^4 & K_{66}^2 + K_{44}^3 + K_{22}^4 \end{bmatrix} \begin{bmatrix} u_7 \\ u_8 \\ u_{10} \end{bmatrix} = \begin{bmatrix} f_3^1 + f_1^2 + f_5^4 \\ f_4^1 + f_2^2 + f_6^4 \\ f_6^2 + f_4^3 + f_2^4 \end{bmatrix} - \begin{bmatrix} K_{7,12}\delta \\ K_{8,12}\delta \\ K_{10,12}\delta \end{bmatrix}$$

4. Compute the FE approximation u^h. Use E=10GPa, v=0.2, d=0.001m and pg=1000N/m2. (*<u>All calculations were performed with HP 50g calculator</u>*)

Firstly, starting with the coordinates and the connectivity matrix, the local stiffness matrix were computed

Coordinates of the local nodes, column: node/ Line: element.

$$X = \begin{matrix} 1.5 & 1.5 & 0 \\ 1.5 & 1.5 & 3 \\ 3 & 3 & 1.5 \\ 3 & 3 & 1.5 \end{matrix} \qquad Y = \begin{matrix} 0 & 1.5 & 0 \\ 1.5 & 0 & 1.5 \\ 0 & 1.5 & 0 \\ 1.5 & 3 & 1.5 \end{matrix}$$

With the coordinates known the areas have been computed, being all the elements equals:

$$A = 1.125\ m$$

*Then the **B** matrix for each element has to be calculated, being equal in elements 1, 3 and 4*

$$B^{(e)} = \frac{1}{2A^{(e)}} \begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix} \qquad b_i^{(e)} = y_j^{(e)} - y_k^{(e)} \qquad c_i^{(e)} = x_k^{(e)} - x_j^{(e)}$$

$b_1 =$ (1.5 0 -1.5) $c_1 =$ (-1.5 1.5 0)

$b_2 =$ (-1.5 0 1.5) $c_2 =$ (1.5 -1.5 0)

$b_3 =$ (1.5 0 -1.5) $c_3 =$ (-1.5 1.5 0)

$b_4 =$ (1.5 0 -1.5) $c_4 =$ (-1.5 1.5 0)

*Then the matrix **D** is computed*

$$D = \begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{21} & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}$$

$d_{11} = d_{22} = E/(1-\nu^2) = 10.41\ e9$

$d_{12} = d_{21} = \nu \cdot d_{11} = 2.08\ e9$

$d_{33} = E/2\cdot(1+\nu) = G = 4.17\ e9$

Being the next step to compute the local stiffness matrices, (in all the elements are equal).

$$K^{(e)} = 10^{9}\cdot \begin{bmatrix} 7.29 & -3.12 & -2.08 & 1.04 & -5.21 & 2.08 \\ & 7.29 & 2.08 & -5.21 & 1.04 & -2.08 \\ & & 2.08 & 0 & 0 & -2.08 \\ & & & 5.21 & -1.04 & 0 \\ & \text{Symmetric} & & & 5.21 & 0 \\ & & & & & 2.08 \end{bmatrix}$$

The body forces for each node of the element were computed being equal distributed in the elements and equal in all of them.

$$f_i = \frac{(At)^{(e)}}{3}\begin{bmatrix} 0 \\ -\rho g \end{bmatrix} \qquad f_i = \begin{bmatrix} 0 \\ -375 \end{bmatrix}$$

Having all the elements the same area, the force vector is equal for all the elements.

Local force contribution → $fl=(0 \quad -375 \quad 0 \quad -375 \quad 0 \quad -375)^T$

Global force vector →

$fg=(0 \quad -375 \quad 0 \quad -1125 \quad 0 \quad -375 \quad 0 \quad -1125 \quad 0 \quad -1125 \quad 0 \quad -375)^T$

Once having all the stiffness and forces matrix computed, the reduced system is computed.

Kgen (reduced) $\cdot 10^9$			fgen (reduced) $\cdot 10^7$
14.58	-3.12	3.31	-1.04
-3.12	14.58	-4.17	-0.0001
3.31	-4.17	14.58	-5.21

Solving the system we can find the unknown displacements:

Node 4 → $U_x = -0.0001\ m$ $U_y = -0.0011\ m$

Node 5 → $U_x = -0.0039\ m$ $U_y = -0\ m$