- Applying the PVW at every element of the mesh,

$$\iint_{A^{(e)}} \delta \bar{\bar{\varepsilon}}^T \bar{\bar{\sigma}}\, t\, dA = \iint_{A^{(e)}} \delta \bar{u}^T \bar{b}\, t\, dA + \oint_{l^{(e)}} \delta \bar{u}^T \bar{t}\, t\, ds + \sum_{i=1}^{6} \delta u_i U_i + \sum_{i=1}^{6} \delta v_i V_i$$

Neglecting the term due to surface forces:

$$[\delta \bar{a}^{(e)}]^T \left[\iint_{A^{(e)}} \bar{\bar{B}}^T \bar{\bar{\sigma}}\, t\, dA - \iint_{A^{(e)}} \bar{\bar{N}}^T \bar{b}\, t\, dA \right] = [\delta \bar{a}^{(e)}]^T \bar{q}^{(e)}$$

$$\iint_{A^{(e)}} \bar{\bar{B}}^T (\bar{\bar{D}} \bar{\bar{B}} a^{(e)} - \bar{\bar{D}} \bar{\bar{\varepsilon}}^0 + \bar{\bar{\sigma}}^0)\, t\, dA - \iint_{A^{(e)}} \bar{\bar{N}}^T \bar{b}\, t\, dA = \bar{q}^{(e)}$$

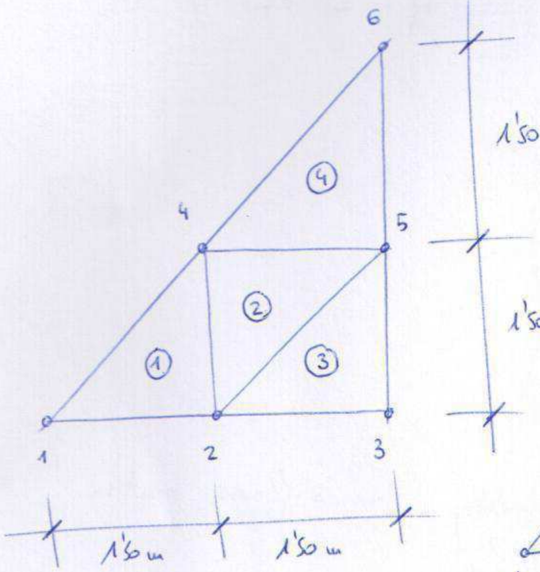
Neglecting the terms due to initial strains and stresses:

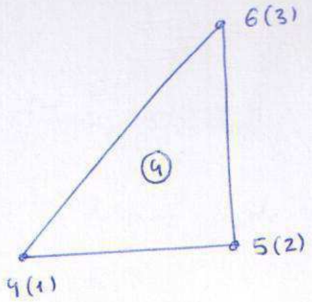
$$\underbrace{\left[\iint_{A^{(e)}} \bar{\bar{B}}^T \bar{\bar{D}} \bar{\bar{B}}\, t\, dA \right]}_{\bar{\bar{K}}^{(e)}} \bar{a}^{(e)} = \underbrace{\iint_{A^{(e)}} \bar{\bar{N}}^T \bar{b}\, t\, dA}_{\bar{f}^{(e)} = \bar{f}_b^{(e)}} + \bar{q}^{(e)}$$

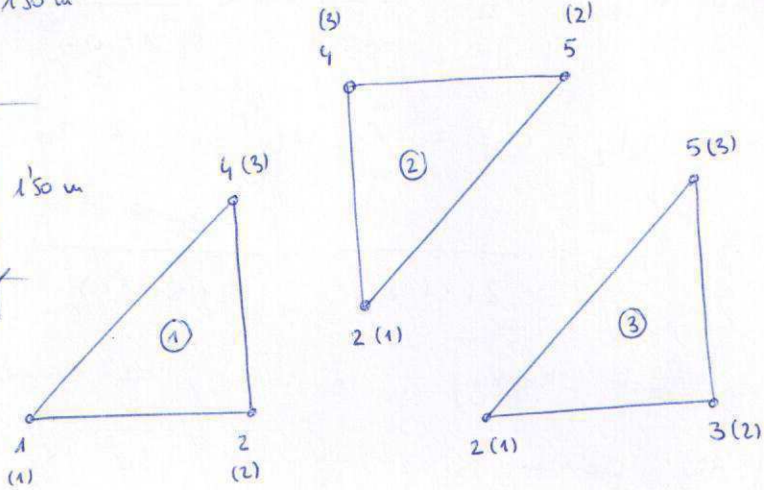
$$\bar{\bar{K}}^{(e)} \cdot \bar{a}^{(e)} = \bar{f}^{(e)} + \bar{q}^{(e)}$$

That are, respectively, the stiffness matrix, the nodal displacement vector, the nodal force vector and the equilibrating nodal (or reaction) forces vector.

• Mesh description of 3-noded triangles :







$$\bar{\bar{K}}_{ij}^{(e)} = \iint_{A^{(e)}} \bar{\bar{B}}_i^T \bar{\bar{D}} \bar{\bar{B}}_j \, t \, dA$$

$$\bar{\bar{K}}_{ij}^{(e)} = \iint_{A^{(e)}} \frac{1}{2A^{(e)}} \begin{bmatrix} b_i & 0 & c_i \\ 0 & c_i & b_i \end{bmatrix} \begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{21} & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \frac{1}{2A^{(e)}} \begin{bmatrix} b_j & 0 \\ 0 & c_j \\ c_j & b_j \end{bmatrix} t \, dA$$

$$\bar{\bar{K}}_{ij} = \left(\frac{t}{4A}\right)^{(e)} \begin{bmatrix} b_i b_j d_{11} + c_i c_j d_{33} & b_i c_j d_{12} + b_j c_i d_{33} \\ c_i b_j d_{21} + b_i c_j d_{33} & b_i b_j d_{33} + c_i c_j d_{22} \end{bmatrix}$$

• Coefficients b_i and c_i for elements 1, 3 and 4 :

$$b_1^{(1)} = b_1^{(3)} = b_1^{(4)} = y_2^{(i)} - y_3^{(i)} = -1'50 \qquad c_1^{(1)} = c_1^{(3)} = c_1^{(4)} = x_3^{(i)} - x_2^{(i)} = 0$$

$$b_2^{(1)} = b_2^{(3)} = b_2^{(4)} = y_3^{(i)} - y_1^{(i)} = 1'50 \qquad c_2^{(1)} = c_2^{(3)} = c_2^{(4)} = x_1^{(i)} - x_3^{(i)} = -1'50$$

$$b_3^{(1)} = b_3^{(3)} = b_3^{(4)} = y_1^{(i)} - y_2^{(i)} = 0 \qquad c_3^{(1)} = c_3^{(3)} = c_3^{(4)} = x_2^{(i)} - x_1^{(i)} = 1'50$$

• Coefficients b_i and c_i for element 2 :

$$b_1^{(2)} = y_2^{(2)} - y_3^{(2)} = 0 \qquad c_1^{(2)} = x_3^{(2)} - x_2^{(2)} = -1'50$$

$$b_2^{(2)} = y_3^{(2)} - y_1^{(2)} = 1'50 \qquad c_2^{(2)} = x_1^{(2)} - x_3^{(2)} = 0$$

$$b_3^{(2)} = y_1^{(2)} - y_2^{(2)} = -1'50 \qquad c_3^{(2)} = x_2^{(2)} - x_1^{(2)} = 1'50$$

- And the components d_{ij} of the constitutive matrix, for a plane stress model, in terms of the values of E and ν are:

$$d_{11} = d_{22} = \frac{E}{1-\nu^2} = \frac{10^{10}}{1-0'2^2}$$

$$d_{12} = d_{21} = \nu \cdot d_{11} = 0'2 \cdot d_{11}$$

$$d_{33} = \frac{E}{2(1+\nu)} = \frac{10^{10}}{2(1+0'2)}$$

- The mesh topology, in order to assembly the stiffness matrix of the problem is:

Element	Nodal connections		
1	1	2	4
2	2	5	4
3	2	3	5
4	4	5	6

- The distribution of the submatrices in the global stiffness matrix is as follows:

$$\bar{\bar{K}} = \begin{bmatrix} \bar{\bar{K}}_{11}^{(1)} & (\bar{\bar{K}}_{12}^{(1)} + \bar{\bar{K}}_{11}^{(2)}) & 0 & \bar{\bar{K}}_{13}^{(1)} & 0 & 0 \\ & (\bar{\bar{K}}_{22}^{(1)} + \bar{\bar{K}}_{11}^{(3)}) & \bar{\bar{K}}_{12}^{(3)} & (\bar{\bar{K}}_{23}^{(1)} + \bar{\bar{K}}_{13}^{(2)}) & (\bar{\bar{K}}_{12}^{(2)} + \bar{\bar{K}}_{13}^{(3)}) & 0 \\ & & \bar{\bar{K}}_{22}^{(3)} & 0 & \bar{\bar{K}}_{23}^{(3)} & 0 \\ & \text{Symmetric} & & (\bar{\bar{K}}_{33}^{(1)} + \bar{\bar{K}}_{33}^{(2)} + \bar{\bar{K}}_{11}^{(4)}) & (\bar{\bar{K}}_{23}^{(2)} + \bar{\bar{K}}_{12}^{(4)}) & \bar{\bar{K}}_{13}^{(4)} \\ & & & & (\bar{\bar{K}}_{22}^{(2)} + \bar{\bar{K}}_{33}^{(3)} + \bar{\bar{K}}_{22}^{(4)}) & \bar{\bar{K}}_{23}^{(4)} \\ & & & & & \bar{\bar{K}}_{33}^{(4)} \end{bmatrix}$$

- Computation of every submatrix in terms of the values of the coefficients b_i, c_i, d_{ij} gives the next elements to be assembled in the global matrix later :

$$\bar{\bar{K}}_{11}^{(1)} = \bar{\bar{K}}_{11}^{(3)} = \bar{\bar{K}}_{11}^{(4)} = 10^{10} \begin{bmatrix} 2'3438 & 0 \\ 0 & 0'9375 \end{bmatrix}$$

$$\bar{\bar{K}}_{13}^{(1)} = \bar{\bar{K}}_{13}^{(3)} = \bar{\bar{K}}_{13}^{(4)} = 10^{10} \begin{bmatrix} 0 & -0'46875 \\ -0'93750 & 0 \end{bmatrix}$$

$$\bar{\bar{K}}_{12}^{(1)} = \bar{\bar{K}}_{12}^{(3)} = \bar{\bar{K}}_{12}^{(4)} = 10^{10} \begin{bmatrix} -2'3438 & 0'4688 \\ 0'9375 & -0'9375 \end{bmatrix}$$

$$\bar{\bar{K}}_{22}^{(1)} = \bar{\bar{K}}_{22}^{(3)} = \bar{\bar{K}}_{22}^{(4)} = 10^{10} \begin{bmatrix} 3'2813 & -1'4063 \\ -1'4063 & 3'2813 \end{bmatrix}$$

$$\bar{\bar{K}}_{23}^{(1)} = \bar{\bar{K}}_{23}^{(3)} = \bar{\bar{K}}_{23}^{(4)} = 10^{10} \begin{bmatrix} -0'9375 & 0'4688 \\ 0'9375 & -2'3438 \end{bmatrix}$$

$$\bar{\bar{K}}_{31}^{(1)} = \bar{\bar{K}}_{31}^{(3)} = \bar{\bar{K}}_{31}^{(4)} = 10^{10} \begin{bmatrix} 0 & -0'93750 \\ -0'46875 & 0 \end{bmatrix} = \bar{\bar{K}}_{13}^{t}$$

$$\bar{\bar{K}}_{33}^{(1)} = \bar{\bar{K}}_{33}^{(3)} = \bar{\bar{K}}_{33}^{(4)} = 10^{10} \begin{bmatrix} 0'9375 & 0 \\ 0 & 2'3438 \end{bmatrix}$$

(with $t = 1m$, and $A = \frac{1'5 \cdot 1'5}{2} = 1'125$)

$$\bar{\bar{K}}_{11}^{(2)} = 10^{10} \begin{bmatrix} 0{,}9375 & 0 \\ 0 & 2{,}3438 \end{bmatrix} \quad , \quad \bar{\bar{K}}_{12}^{(2)} = 10^{10} \begin{bmatrix} 0 & -0{,}93750 \\ -0{,}46875 & 0 \end{bmatrix}$$

$$\bar{\bar{K}}_{13}^{(2)} = 10^{10} \begin{bmatrix} -0{,}9375 & 0{,}9375 \\ 0{,}4688 & -2{,}3438 \end{bmatrix} \quad , \quad \bar{\bar{K}}_{22}^{(2)} = 10^{10} \begin{bmatrix} 2{,}3438 & 0 \\ 0 & 0{,}9375 \end{bmatrix}$$

$$\bar{\bar{K}}_{23}^{(2)} = 10^{10} \begin{bmatrix} -2{,}3438 & 0{,}4688 \\ 0{,}9375 & -0{,}9375 \end{bmatrix} \quad , \quad \bar{\bar{K}}_{33}^{(2)} = 10^{10} \begin{bmatrix} 3{,}2813 & -1{,}4063 \\ -1{,}4063 & 3{,}2813 \end{bmatrix}$$

(with $t = 1$, and $A = 1{,}125$)

While $\bar{\bar{K}}_{ij} = \bar{\bar{K}}_{ji}^{t}$, the global stiffness matrix is symmetric. It is only necessary and sufficient to compute the upper-triangle elements of every element.

- Global stiffness matrix will take this form:

$$\bar{\bar{K}} = \begin{bmatrix} 2{,}3438 & 0 & -1{,}4063 & 0{,}4688 & 0 & 0 & 0 & -0{,}46875 & 0 & 0 & 0 & 0 \\ & 0{,}9375 & 0{,}9375 & 1{,}4063 & 0 & 0 & -0{,}9375 & 0 & 0 & 0 & 0 & 0 \\ & & 5{,}6251 & -1{,}4063 & -2{,}3438 & 0{,}4688 & -1{,}8750 & 1{,}4063 & 0 & -1{,}40625 & 0 & 0 \\ & & & 4{,}2188 & 0{,}9375 & -0{,}9375 & 1{,}4063 & -4{,}6876 & -1{,}40625 & 0 & 0 & 0 \\ & & & & 3{,}2813 & -1{,}4063 & 0 & 0 & -0{,}9375 & 0{,}4688 & 0 & 0 \\ & & & & & 3{,}2813 & 0 & 0 & 0{,}9375 & -2{,}3438 & 0 & 0 \\ & & & & & & 6{,}5626 & -1{,}4063 & -4{,}6876 & 0{,}9376 & 0 & -0{,}46875 \\ & & & & & & & 6{,}5626 & 1{,}8750 & -1{,}8750 & -0{,}9375 & 0 \\ & & \text{symmetric} & & & & & & 6{,}5626 & -1{,}4063 & -0{,}9375 & 0{,}4688 \\ & & & & & & & & & 6{,}5626 & 0{,}9375 & -2{,}3438 \\ & & & & & & & & & & 0{,}9375 & 0 \\ & & & & & & & & & & & 2{,}3438 \end{bmatrix}$$

- The nodal displacement vector is as follows:

$$\bar{a} = \begin{Bmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \bar{a}_3 \\ \bar{a}_4 \\ \bar{a}_5 \\ \bar{a}_6 \end{Bmatrix}$$

, with boundary conditions:

$\bar{a}_1 = \bar{a}_2 = \bar{a}_3 = \bar{0}$ (double support at nodes 1, 2 and 3)

$$\bar{a}_4 = \begin{Bmatrix} u_4 \\ v_4 \end{Bmatrix}$$

$$\bar{a}_5 = \begin{Bmatrix} u_5 \\ v_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ v_5 \end{Bmatrix}$$ (Simple support at node 5, due to the symmetry of the structure)

$$\bar{a}_6 = \begin{Bmatrix} u_6 \\ v_6 \end{Bmatrix} = \begin{Bmatrix} u_6 \\ -d_{01} \end{Bmatrix}$$ (Prescribed displacement at node

which results into the nodal displacement vector:

$$\bar{a} = \{0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad u_4 \quad v_4 \quad 0 \quad v_5 \quad u_6 \quad -d_{01}\}^T$$

- The nodal force vector is composed of the vertical contributions of the self weight ($b_y = -\rho g$) at each node plus the reactions on each node that form the vector $\bar{q}$.

$$\bar{f} + \bar{q} = \begin{Bmatrix} R_{1x} \\ R_{1y} - \rho g \frac{At}{3} \\ R_{2x} \\ R_{2y} - \rho g\, At \\ R_{3x} \\ R_{3y} - \rho g \frac{At}{3} \\ 0 \\ -\rho g\, At \\ R_{5x} \\ -\rho g\, At \\ 0 \\ R_{6y} - \rho g \frac{At}{3} \end{Bmatrix}$$

, being $\bar{f}_{b_i} = \frac{(At)^{(e)}}{3} \begin{Bmatrix} 0 \\ -\rho g \end{Bmatrix}$

$$R_{4x} = R_{4y} = R_{5y} = R_{6x} = 0$$

• Thus, the resulting global system of equations has the following form:

$$\bar{K} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ u_4 \\ v_4 \\ 0 \\ v_5 \\ u_6 \\ -0'01 \end{pmatrix} = \begin{pmatrix} R_{1x} \\ R_{1y} - \rho g \frac{At}{3} \\ R_{2x} \\ R_{2y} - \rho g At \\ R_{3x} \\ R_{3y} - \rho g \frac{At}{3} \\ 0 \\ -\rho g At \\ R_{5x} \\ -\rho g At \\ 0 \\ R_{6y} - \rho g At \end{pmatrix} = \begin{pmatrix} R_{1x} \\ R_{1y} - 375 \\ R_{2x} \\ R_{2y} - 1125 \\ R_{3x} \\ R_{3y} - 375 \\ 0 \\ -1125 \\ R_{5x} \\ -1125 \\ 0 \\ R_{6y} - 375 \end{pmatrix}$$

Which is a 12-ecuations system with 12 unknowns. It can be solved eliminating the rows and columns corresponding to the values on which $a_i = 0$.

This procedure leads to the next results for the nodal displacements:

$u_4 \simeq -0'00042268$ m

$v_4 \simeq -0'00102042$ m

$v_5 \simeq -0'00357147$ m

$u_6 = 0$

And, once obtained, the unknown reactions can be computed:

$R_{1x} \simeq 4'7832 \cdot 10^6$ N $\quad R_{1y} \simeq 3'9630 \cdot 10^6$ N

$R_{2x} \simeq 43'7989 \cdot 10^6$ N $\quad R_{2y} \simeq 41'8902 \cdot 10^6$ N

$R_{3x} \simeq -16'7430 \cdot 10^6$ N $\quad R_{3y} \simeq 83'7084 \cdot 10^6$ N

$R_{5x} \simeq 3'3455 \cdot 10^6$ N

$R_{6y} \simeq 234'3796 \cdot 10^6$ N