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MSc Computational Mechanics

1.
The differential equation can be rewritten as:

$$\frac{\delta^2 u}{\delta^2 x} = f$$

Because of the second order derivative, we need a new function w to be able to solve it. In our case, adding a new function and using integrals is the way to proceed to get the weak form. So:

$$-\int_0^1 w\frac{\delta^2 u}{\delta^2 x}dV = \int_0^1 wf dV$$

Applying the integration by parts rule to the first term of the left hand side of the equation:

$$-\int_0^1 w\frac{\delta^2 u}{\delta^2 x}dV = w\frac{\delta u}{\delta x}\Big|_0^1 + -\int_0^1 w\frac{\delta w}{\delta x}\frac{\delta u}{\delta x}dV$$

The term w is 0 in the Dirichlet boundary conditions. Therefore, the term in red becomes null and we obtain:

$$-\int_0^1 w\frac{\delta^2 u}{\delta^2 x}dV = -\int_0^1 w\frac{\delta w}{\delta x}\frac{\delta u}{\delta x}dV$$

Then, we obtain the weak form of the initial equation:

$$-\int_V w\frac{\delta w}{\delta x}\frac{\delta u}{\delta x}dV = \int_V wf dV \qquad \forall w$$

The finite element approximation of u^h is:

$$u^h = \sum_{j=1}^{n} u_j N_j(x)$$

2.
The linear system of equations to be solved would be extracted from:
With $w_i = N_i$, Einstein notation:
.

$$-\int_V w \frac{\delta N_i}{\delta x}\frac{\delta N_j}{\delta x} dV = \int_V N_i f dV$$

$$K_{ij} = \int_V \frac{\delta N_i}{\delta x}\frac{\delta N_j}{\delta x} dV$$

$$f_{ij} = \int_V N_i f dV$$

Then the system to solve would be:

$$\begin{bmatrix} K_{11} & K_{12} & \dots & K_{1n} \\ K_{21} & K_{22} & \dots & K_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ K_{n1} & K_{n2} & \dots & K_{nn} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

Where K_{ij} is a band matrix and we already know the boundary conditions at the initial and final nodes.

3.

As n=3 , there are 4 nodes and 3 elements. The longitude of the bar is 1. To match the typical matrix numbering, we replace n=4 with i=1,2,...,n The u^h approximation for n=4 is:

$$u^h = \sum_{j=1}^{n} u_j N_j(x) = u_1$$

$$N_1 = \frac{x_2 - x}{l} \qquad N_2 = \frac{x - x_1}{l}$$

$$\frac{dN_1}{dx} = -\frac{1}{l} \qquad \frac{dN_2}{dx} = \frac{1}{l}$$

For each element:

$$K_{ij}^e = \frac{1}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$f_i^1 = \begin{bmatrix} \int_0^{1/3} \frac{x_2 - x}{l} sin(x) \\ \int_0^{1/3} \frac{x - x_1}{l} sin(x) \end{bmatrix} = \begin{bmatrix} 0.01841590961 \\ 0.03662714407 \end{bmatrix}$$

$$f_i^2 = \begin{bmatrix} \int_{1/3}^{2/3} \frac{x_2 - x}{l} sin(x) \\ \int_{1/3}^{2/3} \frac{x - x_1}{l} sin(x) \end{bmatrix} = \begin{bmatrix} 0.0714316275698 \\ -0.0876380579722 \end{bmatrix}$$

$$f_i^3 = \begin{bmatrix} \int_{2/3}^{1} \frac{x_2 - x}{l} sin(x) \\ \int_{2/3}^{1} \frac{x - x_1}{l} sin(x) \end{bmatrix} = \begin{bmatrix} 0.116583715622 \\ 0.129001239292 \end{bmatrix}$$

Calculated with $l^e = 1/3$ because of the described node distribution.
To find the displacements at every node we must solve the following system of equations:

$$3 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ u_2 \\ u_3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0.01841590961 \\ 0.0714316275698 + 0.03662714407 \\ -0.0876380579722 + 0.116583715622 \\ 0.129001239292 \end{bmatrix}$$

Solving:

$$\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1.0272292446 \\ 2.0184388986 \end{bmatrix}$$

Finally, we carry the comparison with the exact result:

$$sin(x) + (3 - sin1)x \Big|_0 = 0$$

$$sin(x) + (3 - sin1)x \Big|_{\frac{1}{3}} = 1.046704369$$

$$sin(x) + (3 - sin1)x \Big|_{\frac{2}{3}} = 2.057389147$$

$$sin(x) + (3 - sin1)x \Big|_1 = 3$$

Obtaining results with low relative error.