$$\boxed{-u'' = f \quad \text{in } [0,1]}$$

$$-\int_0^1 W u'' \, dx = \int_0^1 W f \, dx$$

$$\left\{ \int W u'' dx = \left\{ \begin{array}{ll} a = W & da = W' dx \\ db = u'' dx & b = u' \end{array} \right\} = \left[ W u' \right]_0^1 - \int_0^1 w' u' dx \right\}$$ Integration by parts of the first member

$$\int_0^1 w' u' dx - \left[ W u' \right]_0^1 = \int_0^1 w f \, dx$$

Approximation of the unknown: $\left\{ \begin{array}{l} u \simeq \hat{u} = \sum_{i=0}^{n} N_i u_i \\ u' = \sum_{i=0}^{n} \frac{dN_i}{dx} u_i \end{array} \right\}$

$$\int_0^1 \sum_{i=0}^{n} \frac{dW_i}{dx} \sum_{j=0}^{n} \frac{dN_j}{dx} u_j \, dx - Wu' \Big|_{i=n} + Wu' \Big|_{i=0} = \int_0^1 w f dx$$

Galerkin method : $\{ W_i = N_i \}$

$$\int_0^1 \sum_{i=0}^{n} \frac{dN_i}{dx} \sum_{j=0}^{n} \frac{dN_j}{dx} u_j \, dx = N_n u_n' - N_0 u_0' + \int_0^1 w f dx$$

Due to the discretization of the domain in F.E. : $\left\{ \int_0^1 = \sum_0^{n^{(e)}} \int_{\ell^{(e)}} \right\}$

$$\boxed{\sum_0^{n^{(e)}} \int_{\ell^{(e)}} \frac{dN_i}{dx} \frac{dN_j}{dx} u_j \, dx = N_n u_n' - N_0 u_0' + \sum_0^{n^{(e)}} \int_{\ell^{(e)}} N_i f_i \, dx}$$

$$\bar{\bar{K}}_{ij} = \int_{\ell^{(e)}} \frac{dN_i}{dx} \frac{dN_j}{dx} dx$$

$$\bar{\bar{K}} = \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 & 0 & & \\ & k_{22}^{(1)} + k_{11}^{(2)} & k_{12}^{(2)} & 0 & & \\ & & k_{22}^{(2)} + k_{11}^{(3)} & 0 & & \\ & & & \ddots & & \\ & \text{Symmetric} & & & k_{22}^{(n-1)} + k_{11}^{(n)} & k_{12}^{(n)} \\ & & & & & k_{22}^{(n)} \end{bmatrix}$$

$$\bar{f}_i = \int_{\ell^{(e)}} N_i f_i + q_0 - q_1$$

$$\bar{f} = \begin{bmatrix} f_1^{(1)} + q_0 \\ f_2^{(1)} + f_1^{(2)} \\ f_2^{(2)} + f_1^{(3)} \\ \vdots \\ f_2^{(n-1)} + f_1^{(n)} \\ f_2^{(n)} - q_1 \end{bmatrix}$$

$$\bar{u} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

Discretizing the domain in 3 2-noded elements

$$\bar{u} = \begin{Bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ u_1 \\ u_2 \\ 3 \end{Bmatrix}$$, with $u_0 = 0$ and $u_3 = 3$

$$\bar{\bar{K}} = 3 \cdot \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\bar{f} = \begin{Bmatrix} f_1^{(1)} + q_0 \\ f_2^{(1)} + f_1^{(2)} \\ f_2^{(2)} + f_1^{(3)} \\ f_2^{(3)} + (-q_1) \end{Bmatrix} = \begin{Bmatrix} \int_0^{1/3} N_1^{(1)} \sin x \, dx + q_0 \\ \int_0^{1/3} N_2^{(1)} \sin x \, dx + \int_{1/3}^{2/3} N_1^{(2)} \sin x \, dx \\ \int_{1/3}^{2/3} N_2^{(2)} \sin x \, dx + \int_{2/3}^{1} N_1^{(3)} \sin x \, dx \\ \int_{2/3}^{1} N_2^{(3)} \sin x \, dx - q_1 \end{Bmatrix} =$$

$$= \begin{Bmatrix} -3 \sin 1/3 + 1 + q_0 \\ 3 \sin 1/3 - \cos 1/3 - 3 \sin 2/3 + 3 \sin 1/3 - \cos 1/3 \\ 3 \sin 2/3 - 3 \sin 1/3 - \cos 2/3 - 3 \sin 1 + 3 \sin 2/3 + \cos 2/3 \\ 3 \sin 1 - 3 \sin 2/3 - \cos 1 - q_1 \end{Bmatrix} \simeq \begin{Bmatrix} 0'0184 + q_0 \\ 0'1080 \\ 0'2042 \\ 0'1290 - q_1 \end{Bmatrix}$$

Solving the system, we get the values $\boxed{u_1 = 1'047}$ and $\boxed{u_2 = 2'057}$. Compared with the exact solution given by the ennunciate, they result to be the same.