

Finite Element Method

Homework 1

The given function:

$$-\frac{\partial^2 u}{\partial x^2} = f$$

Boundary conditions:

$$u(0)=0 \text{ and } u(1)=\alpha$$

$$1. \int_0^1 W_i \frac{\partial^2 u}{\partial x^2} dx = \int_0^1 W_i f dx$$

where, W_i is the test function

Integrating by parts,

$$\int_0^1 \left\{ \left[\frac{\partial^2 u}{\partial x^2} \right] \cdot W \right\} dx = W \frac{\partial u}{\partial x} \Big|_0^1 + \int_0^1 \frac{\partial W}{\partial x} \frac{\partial u}{\partial x} dx$$

$$\int_0^1 \left\{ \left[\frac{\partial^2 u}{\partial x^2} \right] \cdot W \right\} dx = \int_0^1 \frac{\partial W}{\partial x} \frac{\partial u}{\partial x} dx$$

$$\int_0^1 W \cdot f dx = \int_0^1 \frac{\partial W}{\partial x} \frac{\partial u}{\partial x} dx$$

Let,

$$U(x) = \sum_{i=0}^1 N_i \cdot u_i$$

$$W(x) = N_i(x)$$

$$\int_0^1 \frac{\partial W}{\partial x} \frac{\partial u}{\partial x} dx = \int_0^1 W \cdot f dx + w \cdot \frac{\partial u}{\partial x} \Big|_0^1$$

Hence, the weak form of the equation is:

$$\int_0^1 \frac{\partial W}{\partial x} \frac{\partial u}{\partial x} dx = \int_0^1 W \cdot f dx + w \cdot \frac{\partial u}{\partial x} \Big|_0^1$$

$$2. \int_0^1 \frac{\partial W_i}{\partial x} \sum_{j=0}^n \frac{\partial N_j}{\partial x} a_j dx = \int_0^1 W_i \cdot f dx + [W_i \cdot R]_1 + [W_i \cdot R]_0$$

$$W_i = N_i$$

$$\int_0^1 \frac{\partial N_i}{\partial x} \sum_{j=0}^n \frac{\partial N_j}{\partial x} a_j dx = \int_0^1 N_i \cdot f dx + [N_i \cdot R]_1 + [N_i \cdot R]_0$$

The global set equation

$$\int_0^1 \frac{\partial W_i}{\partial x} \left(\frac{\partial N_1}{\partial x} a_1 + \frac{\partial N_2}{\partial x} a_2 + \frac{\partial N_3}{\partial x} a_3 + \frac{\partial N_n}{\partial x} a_n \right) dx = \int_0^1 W_i \cdot f dx + [N_i \cdot R]_1 + [N_i \cdot R]_0$$

For i=1

$$\int_0^{1/3} \frac{\partial N_1^{(1)}}{\partial x} \left(\frac{\partial N_1^{(1)}}{\partial x} a_1 + \frac{\partial N_2^{(1)}}{\partial x} a_2 \right) dx = \int_0^{1/3} N_1^{(1)} \cdot f dx + R_0$$

For i=2

$$\int_0^{1/3} \frac{\partial N_2^{(1)}}{\partial x} \left(\frac{\partial N_1^{(1)}}{\partial x} a_1 + \frac{\partial N_2^{(1)}}{\partial x} a_2 \right) dx + \int_{1/3}^{2/3} \frac{\partial N_1^{(2)}}{\partial x} \left(\frac{\partial N_1^{(2)}}{\partial x} a_1 + \frac{\partial N_2^{(2)}}{\partial x} a_2 \right) dx = \int_{1/3}^{2/3} N_1^{(1)} \cdot f dx + \int_0^{1/3} N_1^{(2)} \cdot f dx$$

For i=3

$$\int_{1/3}^{2/3} \frac{\partial N_2^{(2)}}{\partial x} \left(\frac{\partial N_1^{(2)}}{\partial x} a_1 + \frac{\partial N_2^{(2)}}{\partial x} a_2 \right) dx + \int_{1/3}^{2/3} \frac{\partial N_1^{(3)}}{\partial x} \left(\frac{\partial N_1^{(3)}}{\partial x} a_1 + \frac{\partial N_2^{(3)}}{\partial x} a_2 \right) dx = \int_{1/3}^{2/3} N_1^{(1)} \cdot f dx + \int_0^{1/3} N_1^{(2)} \cdot f dx$$

For i=4

$$\int_{2/3}^1 \frac{\partial N_2^{(3)}}{\partial x} \left(\frac{\partial N_1^{(3)}}{\partial x} a_1 + \frac{\partial N_2^{(3)}}{\partial x} a_2 \right) dx = \int_{2/3}^1 N_2^{(3)} \cdot f dx + R_1$$

$$\begin{bmatrix} k_{11} & \cdots & k_{1n} \\ \vdots & \ddots & \vdots \\ k_{n1} & \cdots & k_{nn} \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$$

$$\mathbf{k} \cdot \mathbf{a} = \mathbf{f}$$

Shape functions can be generalized as:

$$N_1^{(e)}(x) = \frac{x_2^{(e)} - x}{l^{(e)}} \quad ; \quad N_2^{(e)}(x) = \frac{x - x_1^{(e)}}{l^{(e)}}$$

$$\frac{\partial N_1^{(1)}}{\partial x} = \frac{-1}{h^{(1)}} \quad \frac{\partial N_1^{(2)}}{\partial x} = \frac{-1}{h^{(2)}} \quad \frac{\partial N_1^{(3)}}{\partial x} = \frac{-1}{h^{(3)}}$$

$$\frac{\partial N_2^{(1)}}{\partial x} = \frac{1}{h^{(1)}} \quad \frac{\partial N_2^{(2)}}{\partial x} = \frac{1}{h^{(2)}} \quad \frac{\partial N_2^{(3)}}{\partial x} = \frac{1}{h^{(3)}}$$

$$k_{11} = \int_0^{1/3} \frac{\partial N_1^{(1)}}{\partial x} \frac{\partial N_1^{(1)}}{\partial x} dx = \frac{1}{h^{2(1)}} \cdot \frac{1}{3} = \frac{1}{3h^{2(1)}} = 3$$

$$k_{12} = \int_0^{1/3} \frac{\partial N_1^{(1)}}{\partial x} \frac{\partial N_2^{(1)}}{\partial x} dx = \frac{1}{3h^{3(2)}} = -3$$

$$k_{22} = \int_0^{1/3} \frac{\partial N_2^{(1)}}{\partial x} \frac{\partial N_2^{(1)}}{\partial x} dx = \frac{1}{3h^{2(2)}} = \frac{1}{3(1/3)^2} = 3$$

$$k_{11} = k_{22}$$

$$k_{12} = k_{21}$$

3. Given,

$$f(x) = \sin(x) ; \alpha = 3$$

$$\begin{aligned} f_1^{(1)} &= \int_0^{1/3} N_1^{(1)} \cdot f(x) dx \\ &= \int_0^{1/3} \frac{x_2^{(1)} - x}{h^{(1)}} \cdot \sin(x) dx = \int_0^{1/3} \frac{x_2^{(1)}}{h^{(1)}} \cdot \sin(x) dx + \int_0^{1/3} \frac{-x}{h^{(1)}} \cdot \sin(x) dx \\ &= -\frac{x_2^{(1)}}{h^{(1)}} \cdot \cos(x) \Big|_0^{1/3} - \frac{1}{h^{(1)}} \cdot (\sin(x) - x \cdot \cos(x)) \Big|_0^{1/3} \\ &= -1 - 3 \cdot \sin(1/3) = 0.0184 \end{aligned}$$

$$f_2^{(1)} = \int_0^{1/3} N_2^{(1)} \cdot f(x) dx = \int_0^{1/3} \frac{x - x_1^{(1)}}{h^{(1)}} \cdot \sin(x) dx$$

$$= 3 \left(\sin\left(\frac{1}{3}\right) - \frac{1}{3} \cdot \cos\left(\frac{1}{3}\right) \right) = 3 \cdot \sin\left(\frac{1}{3}\right) - \cos\left(\frac{1}{3}\right) = 0.0366$$

$$\begin{aligned} f_1^{(2)} &= \int_{\frac{1}{3}}^{\frac{2}{3}} N_1^{(2)} \cdot f(x) dx = \int_{\frac{1}{3}}^{\frac{2}{3}} \frac{x_3 - x}{h} \cdot \sin(x) dx \\ &= -\frac{x_3}{h} \cdot \cos(x) \Big|_{\frac{1}{3}}^{\frac{2}{3}} - \frac{1}{h} \cdot (\sin(x) - x \cdot \cos(x)) \Big|_{\frac{1}{3}}^{\frac{2}{3}} \\ &= -2 \left[\cos\left(\frac{2}{3}\right) - \cos\left(\frac{1}{3}\right) \right] - 3 \left[\sin\left(\frac{2}{3}\right) - \frac{2}{3} \cos\left(\frac{1}{3}\right) - \sin\left(\frac{1}{3}\right) + \frac{1}{3} \cos\left(\frac{1}{3}\right) \right] \\ &= 2\cos\left(\frac{1}{3}\right) - 3\sin\left(\frac{2}{3}\right) + 3\sin\left(\frac{1}{3}\right) - \cos\left(\frac{1}{3}\right) = 0.0714 \end{aligned}$$

$$\begin{aligned} f_2^{(2)} &= \int_{\frac{1}{3}}^{\frac{2}{3}} N_1^{(2)} \cdot f(x) dx = \int_{\frac{1}{3}}^{\frac{2}{3}} \frac{-x_2}{h} \cdot \sin(x) dx + \int_{\frac{1}{3}}^{\frac{2}{3}} \frac{x}{h} \cdot \sin(x) dx \\ &= -\frac{x_2}{h} \cdot \cos(x) \Big|_{\frac{1}{3}}^{\frac{2}{3}} + \frac{1}{h} \cdot (\sin(x) - x \cdot \cos(x)) \Big|_{\frac{1}{3}}^{\frac{2}{3}} \\ &= -\cos\left(\frac{2}{3}\right) - \cos\left(\frac{1}{3}\right) + 3\sin\left(\frac{2}{3}\right) - \left(\frac{2}{3}\right) \cos\left(\frac{2}{3}\right) - \sin\left(\frac{1}{3}\right) + \frac{1}{3} \cos\left(\frac{1}{3}\right) \\ &= 2\cos\left(\frac{1}{3}\right) - \cos\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right) \cos\left(\frac{2}{3}\right) - 3\sin\left(\frac{1}{3}\right) - 2\cos\left(\frac{2}{3}\right) \\ &= \cos\left(\frac{2}{3}\right) + 3\sin\left(\frac{2}{3}\right) - 3\sin\left(\frac{1}{3}\right) = 0.0877 \end{aligned}$$

$$\begin{aligned} f_1^{(3)} &= \int_{\frac{2}{3}}^1 \frac{x_4}{h} \cdot \sin(x) dx + \int_{\frac{2}{3}}^1 \frac{-x}{h} \cdot \sin(x) dx \\ &= -\frac{x_4}{h} \cdot \cos(x) \Big|_{\frac{2}{3}}^1 - \frac{1}{h} \cdot (\sin(x) - x \cdot \cos(x)) \Big|_{\frac{2}{3}}^1 \\ &= -3 \cos(1) + 3 \cos(1) + 3 \cos\left(\frac{2}{3}\right) - 3 \left[\sin(1) - \cos(1) - \sin\left(\frac{2}{3}\right) + \frac{2}{3} \cos\left(\frac{2}{3}\right) \right] \\ &= \cos\left(\frac{2}{3}\right) - 3\sin(1) + 3\sin\left(\frac{2}{3}\right) = 0.1163 \end{aligned}$$

$$\begin{aligned} f_2^{(3)} &= \int_{\frac{2}{3}}^1 \frac{-x_3}{h} \cdot \sin(x) dx + \int_{\frac{2}{3}}^1 \frac{x}{h} \cdot \sin(x) dx \\ &= \frac{x_3}{h} \cdot \cos(x) \Big|_{\frac{2}{3}}^1 + \frac{1}{h} \cdot (\sin(x) - x \cdot \cos(x)) \Big|_{\frac{2}{3}}^1 \\ &= 2 \cos(1) - \cos\left(\frac{2}{3}\right) + 3[\sin(1) - \cos(1)] - \sin\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right) \cos\left(\frac{2}{3}\right) \end{aligned}$$

$$= -\cos(1) + 3\sin(1) - 3\sin\left(\frac{2}{3}\right) = 0.1291$$

The matrix reads

$$R_0 = c, R_1 = d, u = a_2, v = a_3$$

$$3 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ u \\ v \\ 3 \end{bmatrix} = \begin{bmatrix} 0.0184 + c \\ 0.108 \\ 0.2042 \\ 0.1291 + d \end{bmatrix}$$

$$3(2a_2 - a_3) = 0.108 \dots \dots \dots (i)$$

$$3(-a_2 + 2a_3) - 9 = 0.2043 \dots \dots \dots (ii)$$

Now, (i) + (ii) * 2

$$9 a_3 = 18.5164$$

$$a_3 = 2.0573$$

$$6a_2 = 0.108 + 3 a_3$$

$$a_2 = 1.0466$$

exact solution

$$u(x) = \sin x + 3 (\sin 1) x$$

$$\text{for } u(0) = 1.0467$$

$$\text{for } u(1/3) = 1.0467$$

$$\text{for } u(2/3) = 2.0573$$

$$\text{for } u(1) = 3$$

hence,

FEM Solution = Exact Solution