FINITE ELEMENTS
Master of Science in Computational Mechanics/ Numerical Methods
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Homework 2: Plane Elasticity
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Exercise 1· Describe the strong form of the problem in the reduced domain (left half). Indicate accurately the Boundary Conditions in every edge.

The governing/field equations for linear elasticity are:

- $Balance\ equation: \ \nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b} = \rho \ddot{\boldsymbol{u}}.$ $(1.a)$
- $Constitutive\ law: \ \boldsymbol{\sigma} = \boldsymbol{C}: \boldsymbol{\varepsilon} \rightarrow \boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\varepsilon}.$ $(1.b)$
- $Kinematic\ equation: \ \boldsymbol{\varepsilon} = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T).$ $(1.c)$

Where:

- $\boldsymbol{\sigma}$ is the Cauchy stress tensor.
- $\boldsymbol{b}$ is the body foces per unit of volume.
- ρ is the density.
- $\boldsymbol{u}$ is the displacement field.
- $\boldsymbol{C}$ is the stiffness tensor of fourth order, which is simplied to a second order $\boldsymbol{D}$.
- $\boldsymbol{\varepsilon}$ is the strain tensor.

For the boundary conditions we have to take into account that the symmetry plane prevents any displacement on the x direction among it, as well as the fact there is no resultant force perpendicular to the plane.

Finally the strong form is described by:

$$\begin{cases} Balance\ equation: \nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b} = \rho \ddot{\boldsymbol{u}} \\ Dirichlet\ BC: \begin{cases} u = 0, v = 0\ on\ x \in [-3, 0]\ and\ y = 0 \\ u = 0\ on\ x = 0\ y \in [0, 3) \\ u = 0, v = -\delta\ on\ x = 0\ and\ y = 3 \end{cases} \\ Neuman\ BC: \begin{cases} \boldsymbol{\sigma} \cdot \boldsymbol{n} = 0\ on\ x \in [-3, 0]\ and\ y = 3 + x \\ \boldsymbol{\sigma} \cdot \boldsymbol{e}_x = 0\ on\ x = 0\ and\ y \in [0, 3) \end{cases} \end{cases}$$

Exercise 2· Describe the mesh shown in figure 2 by giving the arrays of nodal coordinates **X** and the connectivity matrix **T**. In order to simplify the computations select the local numbering of the nodes such that, in every element, the node in the right angle vertex has loca numer equal to 1.

The vector of nodal coordinates describes the location of the nodes in global numbering. For the case of study:

$$\begin{array}{cc} & \begin{array}{cc} x & y \end{array} \\ \boldsymbol{X} = & \begin{bmatrix} -3 & 0 \\ -1.5 & 0 \\ 0 & 0 \\ -1.5 & 1.5 \\ 0 & 1.5 \\ 0 & 3 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array}$$

where the rows relate with the global numbering of the nodes, and the columns with the X and Y coordinates.

The connectivity matrix is the one which describes the nodes composing an element and also relates the global and local numberings.

Setting the local numbering in counterclockwise direction starting from the right angle, the connectivity matrix is:

$$\begin{array}{cc} & \begin{array}{cccc} ① & ② & ③ & ④ \end{array} \\ \mathrm{T}= & \begin{bmatrix} 2 & 4 & 3 & 5 \\ 4 & 2 & 5 & 6 \\ 1 & 5 & 2 & 4 \end{bmatrix} \begin{array}{c} \mathrm{i} \\ \mathrm{j} \\ \mathrm{k} \end{array} \end{array}$$

where the columns relate with the elements and the rows with the local numbering of those elements.

Exercise 3· Set up the linear system of equations corresponding to the discretization in figure 2. How many degrees of freedom has the system to be solved?

Parting from (1.a) neglecting all the dependencies on z and time, it results the system:

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \sigma_{xy}}{\partial y} + b_x = 0 \\ \dfrac{\partial \sigma_{xy}}{\partial y} + \dfrac{\partial \sigma_y}{\partial x} + b_y = 0 \end{cases} \tag{2}$$

Multiplying by test functions w_1 and w_2 and integrating over all the domain:

$$\begin{cases} \displaystyle\int_\Omega w_1 \frac{\partial \sigma_x}{\partial x} + w_1 \frac{\partial \sigma_{xy}}{\partial y} + w_1 b_x \, d\Omega \\ \displaystyle\int_\Omega w_2 \frac{\partial \sigma_{xy}}{\partial y} + w_2 \frac{\partial \sigma_y}{\partial x} + w_2 b_y \, d\Omega \end{cases} \tag{3}$$

Splitting the integrals and integrating by parts the two first terms of both equations:

$$\begin{cases} -\int_{\Omega} \frac{\partial w_1}{\partial x}\sigma_x d\Omega + \oint_{\Gamma_q} w_1\sigma_x d\Gamma - \int_{\Omega} \frac{\partial w_1}{\partial y}\sigma_{xy} d\Omega + \oint_{\Gamma_q} w_1\sigma_{xy} d\Gamma + \int_{\Omega} w_1 b_x d\Omega \\ -\int_{\Omega} \frac{\partial w_2}{\partial x}\sigma_{xy} d\Omega + \oint_{\Gamma_q} w_2\sigma_{xy} d\Gamma - \int_{\Omega} \frac{\partial w_2}{\partial x}\sigma_y d\Omega + \oint_{\Gamma_q} w_2\sigma_y d\Gamma + \int_{\Omega} w_2 b_y d\Omega \end{cases} \tag{4}$$

Rearranging, putting it in matrix form and considering the traction vector as $\boldsymbol{t} = \boldsymbol{\sigma} \cdot \boldsymbol{n}$:

$$\int_{\Omega} \begin{bmatrix} \frac{\partial w_1}{\partial x} & 0 & \frac{\partial w_1}{\partial y} \\ 0 & \frac{\partial w_2}{\partial y} & \frac{\partial w_2}{\partial x} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} d\Omega = \int_{\Omega} \begin{bmatrix} w_1 b_x \\ w_2 b_y \end{bmatrix} d\Omega + \oint_{\Gamma_q} \begin{bmatrix} w_1 t_x \\ w_2 t_y \end{bmatrix} d\Gamma \tag{5}$$

Substituting the constitutive law (1.b) in Voigt notation and considering plane stress:

$$\int_{\Omega} \begin{bmatrix} \frac{\partial w_1}{\partial x} & 0 & \frac{\partial w_1}{\partial y} \\ 0 & \frac{\partial w_2}{\partial y} & \frac{\partial w_2}{\partial x} \end{bmatrix} \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} & 0 \\ \frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & \frac{E}{2(1+\nu)} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} d\Omega = \int_{\Omega} \begin{bmatrix} w_1 b_x \\ w_2 b_y \end{bmatrix} d\Omega + \oint_{\Gamma_q} \begin{bmatrix} w_1 t_x \\ w_2 t_y \end{bmatrix} d\Gamma \tag{6}$$

Where E is the young modulus and ν the Poisson ratio.

For a 3-noded triangle the interpolation for u^h and v^h is:

$$u^h = N_i u_i = N_1 u_1 + N_2 u_2 + N_3 u_3 \tag{7.a}$$
$$v^h = N_i v_i = N_1 v_1 + N_2 v_2 + N_3 v_3 \tag{7.b}$$

Where:

- $N_i = \frac{1}{2A}(a_i + b_i x + c_i y)$
- $a_i = x_j y_k - x_k y_j$; $b_i = y_j - y_k$; $c_i = x_k - x_j$

with A being the area of the element.

From the kinematic equation (1.c):

- $\varepsilon_x = \frac{\partial u}{\partial x} = \xrightarrow{u^h \approx u} = \frac{\partial N_i u_i}{\partial x}$ (8.a)
- $\varepsilon_y = \frac{\partial v}{\partial y} = \xrightarrow{v^h \approx v} = \frac{\partial N_i v_i}{\partial y}$ (8.b)
- $\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \xrightarrow{u^h \approx u;\ v^h \approx v} = \frac{\partial N_i u_i}{\partial y} + \frac{\partial N_i v_i}{\partial x}$ (8.c)

Substituting (8.a-8.c) into (6) and taking the test functions w_i the same as the shape functions N_i (Galerkin method):

$$\int_\Omega \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_1}{\partial y} \\ 0 & \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} \\ \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_2}{\partial y} \\ 0 & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} \\ \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_3}{\partial y} \\ 0 & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix} \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} & 0 \\ \frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & \frac{E}{2(1+\nu)} \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix} d\Omega$$

$$= \int_\Omega \begin{bmatrix} N_1 b_x \\ N_1 b_y \\ N_2 b_x \\ N_2 b_y \\ N_3 b_x \\ N_3 b_y \end{bmatrix} d\Omega + \oint_{\Gamma_q} \begin{bmatrix} N_1 t_x \\ N_1 t_y \\ N_2 t_x \\ N_2 t_y \\ N_3 t_x \\ N_3 t_y \end{bmatrix} d\Gamma$$

Using index notation:

$$\int_\Omega \frac{1}{2A} \begin{bmatrix} b_i & 0 & c_i \\ 0 & c_i & b_i \end{bmatrix} \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} & 0 \\ \frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & \frac{E}{2(1+\nu)} \end{bmatrix} \frac{1}{2A} \begin{bmatrix} b_j & 0 \\ 0 & c_j \\ c_j & b_j \end{bmatrix} d\Omega \, a_i = \int_\Omega \begin{bmatrix} N_i b_x \\ N_i b_y \end{bmatrix} d\Omega + \oint_{\Gamma_q} \begin{bmatrix} N_i t_x \\ N_i t_y \end{bmatrix} d\Gamma \qquad (9)$$

This last expression can be seen as a linear system of equations of the form **K**ⓔ**a**ⓔ=**f**ⓔ, where

$$k_{ij}^{ⓔ} = \int_\Omega \frac{1}{2A^{ⓔ}} \begin{bmatrix} b_i & 0 & c_i \\ 0 & c_i & b_i \end{bmatrix}^{ⓔ} \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} & 0 \\ \frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & \frac{E}{2(1+\nu)} \end{bmatrix}^{ⓔ} \frac{1}{2A^{ⓔ}} \begin{bmatrix} b_j & 0 \\ 0 & c_j \\ c_j & b_j \end{bmatrix}^{ⓔ} d\Omega \qquad (10)$$

and

$$f_i^{ⓔ} = \int_\Omega \begin{bmatrix} N_i b_x \\ N_i b_y \end{bmatrix}^{ⓔ} d\Omega + \oint_{\Gamma_q} \begin{bmatrix} N_i t_x \\ N_i t_y \end{bmatrix}^{ⓔ} d\Gamma \qquad (11)$$

For the specific case of study we can integrate taking into consideration that all terms inside $k_{ij}^{ⓔ}$ are constant, and that the body force ($\boldsymbol{b} = \rho \boldsymbol{g}$) is uniform among the elements:

$$k_{ij}^{\textcircled{e}} = \frac{1}{4A^{\textcircled{e}}} \begin{bmatrix} b_i & 0 & c_i \\ 0 & c_i & b_i \end{bmatrix}^{\textcircled{e}} \begin{bmatrix} \dfrac{E}{1-\nu^2} & \dfrac{\nu E}{1-\nu^2} & 0 \\ \dfrac{\nu E}{1-\nu^2} & \dfrac{E}{1-\nu^2} & 0 \\ 0 & 0 & \dfrac{E}{2(1+\nu)} \end{bmatrix}^{\textcircled{e}} \begin{bmatrix} b_j & 0 \\ 0 & c_j \\ c_j & b_j \end{bmatrix}^{\textcircled{e}} \tag{12}$$

$$f_i^{\textcircled{e}} = \begin{bmatrix} 0 \\ -\rho g \end{bmatrix} + \oint_{\Gamma_q} \begin{bmatrix} N_i t_x \\ N_i t_y \end{bmatrix}^{\textcircled{e}} d\Gamma \tag{13}$$

The expressions above refer to the elements and therefore they need to be assembled using the connectivity matrix which relates global and nodal numbering.

For element 1:

$$k^{\textcircled{1}} = \begin{array}{cc} \begin{matrix} 2 & \quad 4 & \quad 1 \end{matrix} & \\ \begin{bmatrix} k_{11}^{\textcircled{1}} & k_{12}^{\textcircled{1}} & k_{13}^{\textcircled{1}} \\ k_{21}^{\textcircled{1}} & k_{22}^{\textcircled{1}} & k_{23}^{\textcircled{1}} \\ k_{31}^{\textcircled{1}} & k_{32}^{\textcircled{1}} & k_{33}^{\textcircled{1}} \end{bmatrix} & \begin{matrix} 2 \\ 4 \\ 1 \end{matrix} \end{array} \;;\; f^{\textcircled{1}} = \begin{bmatrix} f_1^{\textcircled{1}} \\ f_2^{\textcircled{1}} \\ f_3^{\textcircled{1}} \end{bmatrix} \begin{matrix} 2 \\ 4 \\ 1 \end{matrix}$$

For element 2:

$$k^{\textcircled{2}} = \begin{array}{cc} \begin{matrix} 4 & \quad 2 & \quad 5 \end{matrix} & \\ \begin{bmatrix} k_{11}^{\textcircled{2}} & k_{12}^{\textcircled{2}} & k_{13}^{\textcircled{2}} \\ k_{21}^{\textcircled{2}} & k_{22}^{\textcircled{2}} & k_{23}^{\textcircled{2}} \\ k_{31}^{\textcircled{2}} & k_{32}^{\textcircled{2}} & k_{33}^{\textcircled{2}} \end{bmatrix} & \begin{matrix} 4 \\ 2 \\ 5 \end{matrix} \end{array} \;;\; f^{\textcircled{2}} = \begin{bmatrix} f_1^{\textcircled{2}} \\ f_2^{\textcircled{2}} \\ f_3^{\textcircled{2}} \end{bmatrix} \begin{matrix} 4 \\ 2 \\ 5 \end{matrix}$$

For element 3:

$$k^{\textcircled{3}} = \begin{array}{cc} \begin{matrix} 3 & \quad 5 & \quad 2 \end{matrix} & \\ \begin{bmatrix} k_{11}^{\textcircled{3}} & k_{12}^{\textcircled{3}} & k_{13}^{\textcircled{3}} \\ k_{21}^{\textcircled{3}} & k_{22}^{\textcircled{3}} & k_{23}^{\textcircled{3}} \\ k_{31}^{\textcircled{3}} & k_{32}^{\textcircled{3}} & k_{33}^{\textcircled{3}} \end{bmatrix} & \begin{matrix} 3 \\ 5 \\ 2 \end{matrix} \end{array} \;;\; f^{\textcircled{3}} = \begin{bmatrix} f_1^{\textcircled{3}} \\ f_2^{\textcircled{3}} \\ f_3^{\textcircled{3}} \end{bmatrix} \begin{matrix} 3 \\ 5 \\ 2 \end{matrix}$$

For element 4:

$$k^{\textcircled{4}} = \begin{array}{cc} \begin{matrix} 5 & \quad 6 & \quad 4 \end{matrix} & \\ \begin{bmatrix} k_{11}^{\textcircled{4}} & k_{12}^{\textcircled{4}} & k_{13}^{\textcircled{4}} \\ k_{21}^{\textcircled{4}} & k_{22}^{\textcircled{4}} & k_{23}^{\textcircled{4}} \\ k_{31}^{\textcircled{4}} & k_{32}^{\textcircled{4}} & k_{33}^{\textcircled{4}} \end{bmatrix} & \begin{matrix} 5 \\ 6 \\ 4 \end{matrix} \end{array} \;;\; f^{\textcircled{4}} = \begin{bmatrix} f_1^{\textcircled{4}} \\ f_2^{\textcircled{4}} \\ f_3^{\textcircled{4}} \end{bmatrix} \begin{matrix} 5 \\ 6 \\ 4 \end{matrix}$$

Where the $k_{ij}^{\textcircled{e}}$ and $f_i^{\textcircled{e}}$ components are to be computed using (10) and (11).

Finally assembling:

$$\boldsymbol{K} = \begin{bmatrix} k_{33}^{\textcircled{1}} & k_{31}^{\textcircled{1}} & 0 & k_{32}^{\textcircled{1}} & 0 & 0 \\ k_{13}^{\textcircled{1}} & k_{11}^{\textcircled{1}} + k_{22}^{\textcircled{2}} + k_{33}^{\textcircled{3}} & k_{31}^{\textcircled{3}} & k_{12}^{\textcircled{1}} + k_{21}^{\textcircled{2}} & k_{32}^{\textcircled{2}} + k_{22}^{\textcircled{3}} & 0 \\ 0 & k_{13}^{\textcircled{3}} & k_{11}^{\textcircled{3}} & 0 & k_{12}^{\textcircled{3}} & 0 \\ k_{23}^{\textcircled{1}} & k_{21}^{\textcircled{1}} + k_{12}^{\textcircled{2}} & 0 & k_{22}^{\textcircled{1}} + k_{11}^{\textcircled{2}} + k_{33}^{\textcircled{4}} & k_{13}^{\textcircled{2}} + k_{31}^{\textcircled{4}} & k_{32}^{\textcircled{4}} \\ 0 & k_{12}^{\textcircled{2}} + k_{23}^{\textcircled{3}} & k_{21}^{\textcircled{3}} & k_{31}^{\textcircled{2}} + k_{13}^{\textcircled{4}} & k_{33}^{\textcircled{2}} + k_{22}^{\textcircled{3}} + k_{11}^{\textcircled{4}} & k_{12}^{\textcircled{4}} \\ 0 & 0 & 0 & k_{23}^{\textcircled{4}} & k_{21}^{\textcircled{4}} & k_{22}^{\textcircled{4}} \end{bmatrix}$$

$$
\boldsymbol{f} = \begin{bmatrix} f_3^{\textcircled{1}} \\ f_1^{\textcircled{1}} + f_2^{\textcircled{2}} + f_3^{\textcircled{3}} \\ f_1^{\textcircled{3}} \\ f_2^{\textcircled{1}} + f_1^{\textcircled{2}} + f_3^{\textcircled{4}} \\ f_3^{\textcircled{2}} + f_2^{\textcircled{3}} + f_1^{\textcircled{4}} \\ f_2^{\textcircled{4}} \end{bmatrix}
$$

Being the vector of unknowns $\boldsymbol{a}$:

$$
\boldsymbol{a} = [a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6]^T \ ; \ with \ a_i = \begin{bmatrix} u_i \\ v_i \end{bmatrix}
$$

In the case of study there are 6 nodes with 2 degrees of freedom each, totaling 12 degrees of freedom. However after applying the Dirichlet Boundary conditions there remain 3 degrees of freedom (u_4, v_4 and v_5).

Exercise 4· Compute the FE approximation u^h. Use $E = 10GPa$, $\nu = 0.2$, $\delta = 10^{-2}$ and $\rho g = 10^3\ N/m^2$

The assembled stiffness matrix $\boldsymbol{K}$ and force vector $\boldsymbol{f}$ take the form:

$$
\boldsymbol{K} = 10^{10} \begin{pmatrix}
0.5208 & 0 & 0.5208 & 0.1042 & 0 & 0 & 0 & 0.1042 & 0 & 0 & 0 & 0 \\
0 & 0.2083 & 0.2083 & -0.2083 & 0 & 0 & -0.2083 & 0 & 0 & 0 & 0 & 0 \\
-0.5208 & 0.2083 & 14.583 & -0.3125 & -0.5208 & 0.1042 & -0.4167 & 0.3125 & 0 & -0.3125 & 0 & 0 \\
0.1042 & -0.2083 & -0.3125 & 14.583 & 0.2083 & -0.2083 & 0.3125 & -10.417 & -0.3125 & 0 & 0 & 0 \\
0 & 0 & -0.5208 & 0.2083 & 0.7292 & -0.3125 & 0 & 0 & -0.2083 & 0.1042 & 0 & 0 \\
0 & 0 & 0.1042 & -0.2083 & -0.3125 & 0.7292 & 0 & 0 & 0.2083 & -0.5208 & 0 & 0 \\
0 & -0.2083 & -0.4167 & 0.3125 & 0 & 0 & 14.583 & -0.3125 & -10.417 & 0.3125 & 0 & -0.1042 \\
-0.1042 & 0 & 0.3125 & -10.417 & 0 & 0 & -0.3125 & 14.583 & 0.3125 & -0.4167 & -0.2083 & 0 \\
0 & 0 & 0 & -0.3125 & -0.2083 & 0.2083 & -10.417 & 0.3125 & 14.583 & -0.3125 & -0.2083 & 0.1042 \\
0 & 0 & -0.3125 & 0 & 0.1042 & -0.5208 & 0.3125 & -0.4167 & -0.3125 & 14.583 & 0.2083 & -0.5208 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.2083 & -0.2083 & 0.2083 & 0.2083 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.1042 & 0 & 0.1042 & -0.5208 & 0 & 0.5208
\end{pmatrix}
$$

$$
\boldsymbol{f} = \begin{pmatrix} R_{1x} \\ R_{1y}\ \text{-375} \\ R_{2x} \\ R_{2y}\ \text{-1125} \\ R_{3x} \\ R_{3y}\ \text{-375} \\ 0 \\ -1125 \\ R_{5x} \\ -1125 \\ R_{6x} \\ R_{6y}\ \text{-375} \end{pmatrix}
$$

And the displacement vector $\boldsymbol{a}$ with the Dirichlet BC remains:

$$\boldsymbol{a} = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad u_4 \quad v_4 \quad 0 \quad v_5 \quad 0 \quad -\delta]^T$$

The only rows with unknowns are 7, 8, and 10; therefore reducing the system:

$$10^{10}\begin{bmatrix} 1.4583 & -0.3125 & 0.3125 \\ -0.3125 & 1.4583 & -0.4167 \\ 0.3125 & -0.4167 & 1.4583 \end{bmatrix}\begin{bmatrix} u_4 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ -1125 \\ -1125 \end{bmatrix} - (-10^{-2})\begin{bmatrix} -0.1042 \\ 0 \\ -0.5208 \end{bmatrix}$$

Solving:

$$\begin{bmatrix} u_4 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} -1.28 \cdot 10^{-4} \\ -1.13 \cdot 10^{-3} \\ -3.86 \cdot 10^{-3} \end{bmatrix} m$$

The reactions can be computed *'a posteriori'*:

$$\boldsymbol{R} = 10^3\begin{bmatrix} 1179 \\ 267 \\ 9081 \\ 11398 \\ -4028 \\ 20144 \\ 0 \\ 0 \\ -534 \\ 0 \\ -5698 \\ -31805 \end{bmatrix} N$$

Recalling expressions (7.a) and (7.b), $\boldsymbol{u}^h$ is interpolated using global shape functions and nodal solutions.

$$u^h = N_i u_i$$
$$v^h = N_i v_i$$

For i ranging from 1 to 6.