# FINITE ELEMENTS

**Master of Science in Computational Mechanics/ Numerical Methods**

**Fall Semester 2015**

Homework 1: Basics of FE

November 10$^{rd}$, 2015

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Consider the following differential equation

$$-u'' = f \; in \; [0,1]$$

with the boundary conditions $u(0) = 0$ and $u(1) = \alpha$

The Finite Element discretization is a 2-noded linear mesh given by the nodes $x_i = ih$ for $i = 0,1,\ldots,n$ and $h = 1/n$.

1. Find the weak form of the problem. Describe the FE approximation $u^h$.
2. Describe the linear system of equations to be solved
3. Compute the FE approximation $u^h$ for $n = 3$, $f(x) = \sin x$ and $\alpha = 3$. Compare it with the exact solution, $u(x) = \sin x + (3 - \sin 1)x$.

1. Being $A(u)$ the differential govern equation and $B(u)$ the equation containing the boundary conditions:

$$A(u) = \frac{d^2u}{dx^2} + f = 0 \quad in \; \omega$$

$$B(u) = \begin{cases} u(0) = 0 & in \; \Gamma_u \\ u(1) = \alpha & in \; \Gamma_u \end{cases}$$

We can multiply the expressions above for an arbitrary weighting function w(x) and integrate over each domain

$$\int_\Omega w(x)A(u)d\Omega + \int_\Gamma \overline{w}(x)B(u)d\Gamma$$

Taking into consideration that there are no Neuman boundary conditions defined ($\Gamma_q = \emptyset$) and that the Dirichlet conditions are imposed when solving the resulting system of equations, the second integral becomes 0. Developing:

$$\int_\Omega w\left[\frac{d^2u}{dx^2} + f\right]d\Omega = \int_\Omega w\,\frac{d^2u}{dx^2}d\Omega + \int_\Omega w\,f d\Omega$$

Integrating by parts the first term; ($\int_0^l udv = [uv]_0^l - \int_0^l vdu$) being $u = w$ and $v = du/dx$

$$-\int_\Omega \frac{dw}{dx}\frac{du}{dx}d\Omega + \int_\Gamma w\frac{du}{dx}d\Gamma + \int_\Omega w\,fd\Omega$$

$$-\int_\Omega \frac{dw}{dx}\frac{du}{dx}d\Omega + \left[w\frac{du}{dx}\right]_0^1 + \int_\Omega w\,fd\Omega$$

By doing this we got rid of the second derivatives while keeping an expression (called weak form) that has the same solutions as the original differential equation.

The finite element method approximation to be used is of the form:

$$u \cong u^h = \sum_{i=1}^{n} N_i(x)a_i$$

Where $N_i(x)$ are the interpolation functions and $a_i$ the solution on the nodes. The interpolation functions are defined locally inside the elements as:

$$N_1^{ⓔ}(x) = \frac{x_2^{ⓔ} - x}{l^{ⓔ}}$$

$$N_2^{ⓔ}(x) = \frac{x - x_1^{ⓔ}}{l^{ⓔ}}$$

Putting this back in the weak form

$$-\int_\Omega \frac{dw}{dx}\frac{du^h}{dx}d\Omega + \int_\Gamma w\frac{du^h}{dx}d\Gamma + \int_\Omega w\,fd\Omega$$

2. Substituting the interpolation functions in the weak form:

$$-\int_\Omega \frac{dw_i}{dx}\frac{dN_j}{dx}a_j\,d\Omega + [w_i\,q_i]_0^1 + \int_\Omega w_i\,f_i d\Omega$$

As $a_j$ is an independent scalar it can be put out of integral. Then rearranging we obtain

$$\int_\Omega \frac{dw_i}{dx}\frac{dN_j}{dx}d\Omega\ a_j = [w_i\,q]_0^1 + \int_\Omega w_i\,fd\Omega$$

We can see that now it takes the form of a linear system in matrix form ($\boldsymbol{Ka} = \boldsymbol{f}$), where

$$K_{ij} = \int_\Omega \frac{dw_i}{dx}\frac{dN_j}{dx}d\Omega$$

$$f_i = [w_i\,q]_0^1 + \int_\Omega w_i\,fd\Omega$$

We can also see that if we take the weighting function the same as the interpolation function, the stiffness matrix becomes symmetric. This is called Galerkin method and the stiffness matrix and nodal force vector are now:

$$K_{ij} = \int_\Omega \frac{dN_i}{dx}\frac{dN_j}{dx}d\Omega$$

$$f_i = [N_i\,q]_0^1 + \int_\Omega N_i\,fd\Omega$$

3. For solving the problem we need to obtain the global stiffness matrix and vector of nodal forces. It is necessary then to map the relation between the global and local interpolating functions:

| Interval | Global | Local | Interval | Global | Local |
|---|---|---|---|---|---|
| $x \in [0, 1/3]$ | $N_1$ | $N_1^{①}$ | $x \in [1/3, 2/3]$ | $N_3$ | $N_2^{②}$ |
| $x \notin [0, 1/3]$ | $N_1$ | 0 | $x \in [2/3, 1)$ | $N_3$ | $N_1^{③}$ |
| $x \in [0, 1/3)$ | $N_2$ | $N_2^{①}$ | $x \notin [1/3, 1]$ | $N_3$ | 0 |
| $x \in [1/3, 2/3]$ | $N_2$ | $N_1^{②}$ | $x \in [2/3, 1]$ | $N_4$ | $N_2^{③}$ |
| $x \notin [0, 2/3]$ | $N_2$ | 0 | $x \notin [2/3, 1]$ | $N_4$ | 0 |

For computing the global stiffness matrix, we need to compute the local ones. In the particular case of this problem we can see that all the local stiffness matrix have the same expression, which is:

$$\int_0^{l^{ⓔ}} \frac{dN_i}{dx}\frac{dN_j}{dx} d\Omega$$

$$\int_0^{l^{ⓔ}} \frac{dN_i}{dx}\frac{dN_j}{dx} d\Omega$$

$$\frac{dN_1}{dx} = -\frac{1}{l^{ⓔ}}\ ;\ \frac{dN_2}{dx} = \frac{1}{l^{ⓔ}}$$

$$l^{①} = l^{②} = l^{③}$$

$$\begin{bmatrix} K_{11} & k_{12} \\ K_{21} & k_{22} \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$$

The nodal forces vector is computed directly on global reference taking into account the table for the splitting of the integrals.

The general form of the vector is

$$f_i = [N_i\, q]_0^1 + \int_\Omega N_i\, f d\Omega$$

Evaluating for all the nodes:

$$f_1 = q_1 + \int_\Omega N_1\, f d\Omega = q_1 + \int_0^{1/3} \frac{1/3 - x}{1/3}\sin(x)\ d\Omega = q_1 + 0.018416$$

$$f_2 = \int_\Omega N_2\, f d\Omega = \int_0^{1/3} \frac{x - 0}{1/3}\sin(x)\ d\Omega + \int_{1/3}^{2/3} \frac{2/3 - x}{1/3}\sin(x)\ d\Omega = 0.108059$$

$$f_3 = \int_\Omega N_3\, f d\Omega = \int_{1/3}^{2/3} \frac{x - 1/3}{1/3}\sin(x)\ d\Omega + \int_{2/3}^{1} \frac{1 - x}{1/3}\sin(x)\ d\Omega = 0.204218$$

$$f_4 = q_4 + \int_\Omega N_4\, f d\Omega = q_4 + \int_{2/3}^{1} \frac{x - 2/3}{1/3}\sin(x)\ d\Omega = q_4 + 0.01290$$

Assembling the stiffness matrix the system to solve in matrix form is:

$$\begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 3+3 & -3 & 0 \\ 0 & -3 & 3+3 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} q_1 + 0.018416 \\ 0.108059 \\ 0.20421 \\ q_4 + 0.1290 \end{bmatrix}$$

Substituting the Dirichlet boundary conditions ($u_1 = 0$, $u_4 = \alpha = 3$), we can solve the reduced system

$$\begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0.108059 \\ 0.20421 + 9 \end{bmatrix},$$

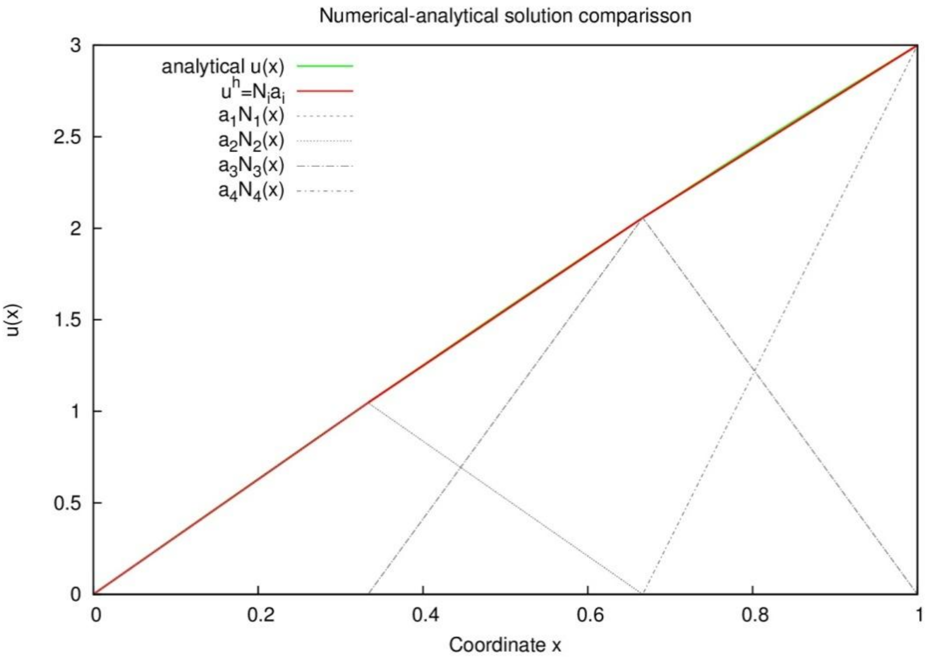
obtaining

$$\boxed{u_2 = 1.0467;\ \ u_3 = 2.05739}$$

The reactions flux can be computed a posteriori once the solution on the nodes is known.

$$\boxed{q_1 = -3.1585;\ q_4 = 2.6988}$$

Plotting the analytical solution $u(x) = \sin x + (3 - \sin 1)x$ and the approximate fem solution $u^h(x) = \sum_{i=1}^{n} N_i(x) a_i$




we can see that for the solved case using 3 linear elements the approximate solution obtained seems practically the exact one given by the analytical expression, thus we can say that the solution is accurate.