Consider the following differential equation

$$-\frac{d^2u}{dx^2} = \mathrm{f} \quad in \quad]0,1[$$

with the boundary conditions:

$$\begin{cases} u(0) = 0 \\ u(1) = \alpha \end{cases}$$

The Finite Element discretization is a 2-noded linear mesh given by the nodes $x_i = ih$ for $i = 0, 1, \dots , n$ and $h = 1/n$.

1. Find the weak form of the problem. Describe the FE approximation u^h.
2. Describe the linear system of equation to be solved.
3. Compute the FE approximation u^h for $n = 3$, $Q(x) = \sin x$ and $\alpha = 3$. Compute it with the exact solution $u(x) = \sin x + (3 - \sin 1)x$.

1. Find the weak form of the problem. Describe the FE approximation u^h.

So, we have:

- The governing differential equation:

$$-\frac{d^2u}{dx^2} = \mathrm{f} \quad in \quad]0,1[\qquad (1)$$

- And the boundary conditions:

$$\begin{cases} u(0) = 0 \\ u(1) = \alpha \end{cases}$$

in the boundary Γ of Ω.

To find the weak form of this problem we proceed as follows:

We multiply (1) by an arbitrary w(x) weighting function

$$-w(x)\frac{d^2u}{dx^2} = \mathrm{f\,w(x)}$$

Such that w(x) is 0 in Γ

and then we integrate over the domain:

$$-\int_0^1 w(x)\frac{d^2u}{dx^2}\mathrm{dx} = \int_0^1 f\ \mathrm{w(x)dx}$$

Remembering the integration by parts formula:

$$\int_a^b f dg + \int_a^b g df = [fg]\,\begin{matrix} b \\ a \end{matrix}$$

In our case a=0, b=1, g=w and

$$df = \frac{d^2u}{dx^2}$$

And

$$\int_0^1 w(x)\frac{d^2u}{dx^2}\mathrm{dx} = \left[\frac{du}{dx}\ w(x)\right]_0^1 - \int_0^1 \frac{du}{dx}\frac{dw}{dx}\ dx$$

$$\left[\frac{du}{dx}\,w(x)\right]_0^1 = 0$$

because we have defined w(x) such that w(x)=0 in Γ, and

$$-\int_0^1 w(x)\frac{d^2u}{dx^2}\mathrm{dx} = \int_0^1 \frac{du}{dx}\frac{dw}{dx}\,dx$$

So substituting:

$$\int_0^1 \frac{du}{dx}\frac{dw}{dx}\,dx = \int_0^1 f\;\mathrm{w(x)dx} \quad (2)$$

We have found the **<u>weak form of the problem</u>**.

In order to approximate the algebraic equation by a numeric one, we express u as a sum of n products of linear combination of products of a_j (unknown) and $N_j(x)$ (a shape function such each of them is 1 when j=n and 0 in any j≠n)

So, we would have:

$$u \approx u^h = \sum_{j=1}^{n} N_j a_j \; = \sum_{j=1}^{n} a_j \mathrm{Sin}(\frac{x_j \pi}{2l})$$

$$N_j = \mathrm{Sin}\left(\frac{x_j \pi}{l}\right)$$

And now we just substitute this approximation $u \approx u^h$ in (2):

$$\int_0^1 \frac{d}{dx}(\sum_{j=1}^{n} N_j a_j \,) \; \frac{dw}{dx} \; dx \; = \int_0^1 f \;\; \mathrm{w(x)dx}$$

Next step is to choose a suitable weight function w. We finally choose

$$w = W_i(\mathrm{x}) = N_i(x) \;\; \begin{cases} 1 \; when \; i = n \\ 0 \;\; when \; i \neq n \end{cases}$$

known as Galerkin method. So now:

$$\int_0^1 \frac{d}{dx}(\sum_{j=1}^{n} N_j a_j \,) \; \frac{d(N_i(x))}{dx} \; dx \; = \int_0^1 f \;\; N_i(x) \mathrm{dx}$$

$$\int_0^1 \frac{d}{dx}(\sum_{j=1}^{n} N_j a_j \,) \; \frac{d}{dx}(N_1(x)) \, dx \; = \int_0^1 f \;\; N_1(x) \; \mathrm{dx}$$

$$\int_0^1 \frac{d}{dx}(N_1 a_1 + \; N_2 a_2 + \cdots + N_n a_n) \; \frac{d}{dx}(N_1(x)) \, dx \; = \int_0^1 f \;\; N_1(x) \; \mathrm{dx}$$

And this last equation has the following form:

Ka=f

$$K=\begin{pmatrix} \int_0^1 \frac{d}{dx}(N_1 a_1)\frac{d(N_1(x))}{dx}dx & \cdots & \int_0^1 \frac{d}{dx}(N_n a_n)\frac{d(N_1(x))}{dx}dx \\ \vdots & \ddots & \vdots \\ \int_0^1 \frac{d}{dx}(N_1 a_n)\frac{d(N_n(x))}{dx}dx & \cdots & \int_0^1 \frac{d}{dx}(N_n a_n)\frac{d(N_n(x))}{dx}dx \end{pmatrix}$$

But we will use $Kij = (\frac{j\pi}{l})^2 \int_0^l W_{i(x)} \cdot \text{Sin}(\frac{j\pi x}{l})dx$

$$f_i = \int_0^l f \; W_i(x)dx$$

$$\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} \int_0^1 f \; W_1(x)dx \\ \vdots \\ \int_0^1 f \; W_n(x)dx \end{pmatrix}$$

In our problem we have a 2-noded linear mesh with n nodes x_i , such that

x_i = ih for i = 0, 1, … , n and h = 1/n

If we are asked for this particular case: u^h for $n = 3$, $f(x) = \sin x$ and $\alpha = 3$, then:

	<u>Exact solution</u>
$X_0=0$	u(0)=0
$X_1=1\frac{1}{3} = \frac{1}{3}$	u(1/3)=1
$X_2=2\frac{1}{3} = \frac{2}{3}$	u(2/3)=2
$X_3=3\frac{1}{3} = 1$	u(1)=3

With the boundary conditions:

$$\begin{cases} u(0) = 0 \\ u(1) = 3 \end{cases}$$

So, $u^h = \sum_{j=1}^{n} N_j a_j$

And we have choosen N_j:

$$N_j = \text{Sin}\left(\frac{x_j \pi}{l}\right)$$

0 <x< l, with l=1

in order to satisfy

$$w = W_i(\text{x}) = N_i(x) \begin{cases} 1 \; when \; i = n \\ 0 \; \; when \; i \neq n \end{cases}$$

Note we will use:

$$\int Sin(ax)Sin(bx)dx = -\frac{\sin(a+b)x}{2(a+b)} + \frac{\sin(a-b)\,x}{2(a-b)} + C$$

And l=1,

$f_1 = \int_0^{1/3} sin\ x\ \sin(\pi x)dx == -\frac{\text{Sin}(1+\pi)(\frac{1}{3})}{2(1+\pi)} + \frac{\text{Sin}(1-\pi)(\frac{1}{3})}{2(1-\pi)} + \frac{\sin 0}{2(a+b)} - \frac{\sin(0)}{2(a-b)} = -0.118 + 0.1529 \approx 0{,}2714$

$$K_{11} = \pi^2 \int_0^1 W_1(\text{x})\, s\,\text{in}(\pi x)\, dx = \pi^2 \int_0^1 \sin(\pi x)\sin(\pi x)\ \ dx = \frac{\pi^2}{2} \approx 4.9348$$

$f_2 = \int_0^{1/3} sin\ x\ \sin(2\pi x)dx == -\frac{\text{Sin}(1+2\pi)(\frac{1}{3})}{2(1+2\pi)} + \frac{\text{Sin}(1-2\pi)(\frac{1}{3})}{2(1-2\pi)} = -0{,}045 + 0.1667 \approx 0{,}1217$

$f_3 = \int_0^{1/3} sin\ x\ \sin(3\pi x)dx == -\frac{\text{Sin}(1+3\ \ )(\frac{1}{3})}{2(1+3\pi)} + \frac{\text{Sin}(1-3\pi)(\frac{1}{3})}{2(1-3\pi)} =$ -0.6667+ 0.0194$\approx -0{,}6473$

$$K_{12} = 4\pi^2 \int_0^1 W_1(\mathrm{x})\, Sin(2\pi x)dx = 4\pi^2 \int_0^1 \mathrm{Sin}(\pi x)\ Sin(2\pi x)dx$$
$$= 4\pi^2 \left(-\frac{\mathrm{Sin(3\pi)}}{(\,6\pi\,)} + \frac{\mathrm{Sin(-\pi)}}{-2\pi} \right) = 0$$

$$K_{21} = \pi^2 \int_0^1 W_2(\mathrm{x})\, \mathrm{Sin}(\pi x)\, dx = \pi^2 \int_0^1 \mathrm{Sin}(2\pi x)\, \mathrm{Sin}(\pi x)\, dx = 0$$

$$K_{22} = 4\pi^2 \int_0^1 W_2(\mathrm{x})\, Sin(2\pi x)dx = 4\pi^2 \int_0^1 \mathrm{Sin}(2\pi x)\ Sin(2\pi x)dx = 2\pi^2$$
$$\approx 19.7392$$

$$K_{33} = 9\pi^2 \int_0^1 W_3(\mathrm{x})\, Sin(3\pi x)dx = 9\pi^2 \int_0^1 \mathrm{Sin}(3\pi x)\ Sin(3\pi x)dx = \frac{9\pi^2}{2} = 44.4132$$

$K_{31} = 0$

$$K_{13} = 0$$

$$K_{32} = 0$$

$$K_{23} = 0$$

$$\begin{pmatrix} 4.9348 & 0 & 0 \\ 0 & 19.7392 & 0 \\ 0 & 0 & 44.4132 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0{,}2714 \\ 0{,}1217 \\ -0{,}6473 \end{pmatrix}$$

$$a_1 \approx 0.0550$$

$$a_2 \approx 0.0062$$

$$a_3 \approx -0.0146$$

U(x) = $a_1\ N_1(x) + \ a_2\ N_2(x) + \ a_3\ N_3(x)$

U(x)=0.055 Sin(πx) + 0.0062Sin(2πx) -0.0146 Sin(3πx)

U(0)=0

U(1/3)= 0.0476 + 0.0054 – 0 = 0.053

U(2/3)=

U(1) = 0

These are the numeric solutions. And they should be close to those exact solutions written in page 5.

So I have some mistakes…