Consider the following differential equation

$$-\frac{d^2u}{dx^2} = \mathrm{f} \quad in \quad]0,1[$$

with the boundary conditions:

$$\begin{cases} u(0) = 0 \\ u(1) = \alpha \end{cases}$$

The Finite Element discretization is a 2-noded linear mesh given by the nodes $x_i = ih$ for $i = 0, 1, \dots , n$ and $h = 1/n$.

1. Find the weak form of the problem. Describe the FE approximation u^h.
2. Describe the linear system of equation to be solved.
3. Compute the FE approximation u^h for $n = 3$, $Q(x) = \sin x$ and $\alpha = 3$. Compute it with the exact solution $u(x) = \sin x + (3 - \sin 1)x$.

1. Find the weak form of the problem. Describe the FE approximation u^h.

So, we have:

- The governing differential equation:

$$-\frac{d^2u}{dx^2} = \mathrm{f} \quad in \quad]0,1[\qquad (1)$$

- And the boundary conditions:

$$\begin{cases} u(0) = 0 \\ u(1) = \alpha \end{cases}$$

in the boundary Γ of Ω.

To find the weak form of this problem we proceed as follows:

We multiply (1) by an arbitrary w(x) weighting function

$$-w(x)\frac{d^2u}{dx^2} = \mathrm{f\,w(x)}$$

Such that w(x) is 0 in Γ

and then we integrate over the domain:

$$-\int_0^1 w(x)\frac{d^2u}{dx^2}\,\mathrm{dx} = \int_0^1 f\ \ \mathrm{w(x)dx}$$

Remembering the integration by parts formula:

$$\int_a^b f dg + \int_a^b g df = [fg]\,\begin{matrix} b \\ a \end{matrix}$$

In our case a=0, b=1, g=w and

$$df = \frac{d^2u}{dx^2}$$

And

$$\int_0^1 w(x)\frac{d^2u}{dx^2}\,\mathrm{dx} = \left[\frac{du}{dx}\ w(x)\right]_0^1 - \int_0^1 \frac{du}{dx}\,\frac{dw}{dx}\ dx$$

$$\left[\frac{du}{dx}\ w(x)\right]_0^1 = 0$$

because we have defined w(x) such that w(x)=0 in Γ, and

$$-\int_0^1 w(x)\frac{d^2u}{dx^2}\mathrm{dx} = \int_0^1 \frac{du}{dx}\frac{dw}{dx}\ dx$$

So substituting:

$$\int_0^1 \frac{du}{dx}\frac{dw}{dx}\ dx\ = \int_0^1 f\ \ \mathrm{w(x)dx}\quad (2)$$

We have found the **weak form of the problem**.

In order to approximate the algebraic equation by a numeric one, we express u as a sum of n products of linear combination of products of a_j (unknown) and $N_j(x)$ (a shape function such each of them is 1 when j=n and 0 in any j≠n)

So, we would have:

$$u \approx u^h = \sum_{j=1}^{n} N_j a_j = \sum_{j=1}^{n} a_j \mathrm{Sin}(\frac{x_j \pi}{2l})$$

$$N_j = \mathrm{Sin}\left(\frac{x_j \pi}{l}\right)$$

And now we just substitute this approximation $u \approx u^h$ in (2):

$$\int_0^1 \frac{d}{dx}(\sum_{j=1}^{n} N_j a_j)\ \frac{dw}{dx}\ dx = \int_0^1 f\ \mathrm{w(x)dx}$$

Next step is to choose a suitable weight function w. We finally choose

$$w = W_i(\mathrm{x}) = N_i(x)\ \begin{cases} 1\ when\ i = n \\ 0\ \ when\ i \neq n \end{cases}$$

known as Galerkin method. So now:

$$\int_0^1 \frac{d}{dx}(\sum_{j=1}^{n} N_j a_j)\ \frac{d(N_i(x))}{dx}\ dx = \int_0^1 f\ N_i(x)\mathrm{dx}$$

$$\int_0^1 \frac{d}{dx}(\sum_{j=1}^{n} N_j a_j)\ \frac{d}{dx}(N_1(x))\ dx = \int_0^1 f\ N_1(x)\ \mathrm{dx}$$

$$\int_0^1 \frac{d}{dx}(N_1 a_1 + N_2 a_2 + \cdots + N_n a_n)\ \frac{d}{dx}(N_1(x))\ dx = \int_0^1 f\ N_1(x)\ \mathrm{dx}$$

And this last equation has the following form:

Ka=f

$$K=\begin{pmatrix} \int_0^1 \frac{d}{dx}(N_1 a_1)\frac{d(N_1(x))}{dx}dx & \cdots & \int_0^1 \frac{d}{dx}(N_n a_n)\frac{d(N_1(x))}{dx}dx \\ \vdots & \ddots & \vdots \\ \int_0^1 \frac{d}{dx}(N_1 a_n)\frac{d(N_n(x))}{dx}dx & \cdots & \int_0^1 \frac{d}{dx}(N_n a_n)\frac{d(N_n(x))}{dx}dx \end{pmatrix}$$

$$f_i = \int_0^l f \ W_i(x)dx$$

$$\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} \int_0^1 f \ W_1(x)dx \\ \vdots \\ \int_0^1 f \ W_n(x)dx \end{pmatrix}$$

In our problem we have a 2-noded linear mesh with n nodes x_i , such that x_i = ih for i = 0, 1, … , n and h = 1/n

If we are asked for this particular case: u^h for *n = 3*, *f(x) = sin x* and $\alpha = 3$, then:

<u>Exact solution</u> (algebraic solution)

$X_0=0$	u(0)=0
$X_1=1\frac{1}{3} = \frac{1}{3}$	u(1/3)=1
$X_2=2\frac{1}{3} = \frac{2}{3}$	u(2/3)=2
$X_3=3\frac{1}{3} = 1$	u(1)=3

With the boundary conditions:

$$\begin{cases} u(0) = 0 \\ u(1) = 3 \end{cases}$$

So, $u^h = \sum_{j=1}^{n} N_j a_j$

And we have chosen N_j:

$$N_j = 3\text{x}$$

0 <x< l, with l=1

in order to satisfy

$$w = W_i(\text{x}) = N_i(x) \ \begin{cases} 1 \ when \ i = n \\ 0 \ \ when \ i \neq n \end{cases}$$

$$f_1 \ = \int_0^{1/3} 3x \ sin \ x \ \ dx \approx -0{,}9825$$

(Integrating by parts)

$$f_2 \ = \int_{1/3}^{2/3} sin \ x \ \ (3x)dx \approx 0{,}5004$$

$$f_3 \ = \int_{\frac{2}{3}}^{1} sin \ x \ \ (3x)dx \approx 2{,}9475$$

$$K_{12} = \int_0^1 \frac{d}{dx}(3\text{x}) \ \frac{d}{dx}(3\text{x}) \, dx = \ 3$$

$$K_{21} = \int_0^1 \frac{d}{dx}(3\text{x}) \ \frac{d}{dx}(3\text{x}) \, dx = \ -3$$

$$K_{22} = \int_0^1 \frac{d}{dx}(3\text{x}) \ \frac{d}{dx}(3\text{x}) \, dx = \ 6$$

$$K_{33} = \int_0^1 \frac{d}{dx}(3\text{x}) \ \frac{d}{dx}(3\text{x}) \, dx = \ 9$$

$$K_{31} = \int_0^1 \frac{d}{dx}(3\text{x})\ \frac{d}{dx}(3\text{x})\, dx = -6$$

$$K_{13} = \int_0^1 \frac{d}{dx}(3\text{x})\ \frac{d}{dx}(3\text{x})\, dx = 6$$

$$K_{23} = \int_0^1 \frac{d}{dx}(3\text{x})\ \frac{d}{dx}(3\text{x})\, dx = 3$$

$$K_{32} = \int_0^1 \frac{d}{dx}(3\text{x})\ \frac{d}{dx}(3\text{x})\, dx = -3$$

$$3\begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} -0{,}9825 \\ 0{,}5004 \\ 2{,}9475 \end{pmatrix}$$

And we must solve this system of three equations with three unknowns:

$$a_1 \approx 0.4085$$

$$a_2 \approx 0.087$$

$$a_3 \approx 0{,}739$$

U(x) = $a_1\ N_1(x) +\ a_2\ N_2(x) +\ a_3\ N_3(x)$

U(x)=0.4085 (3x) + 0.087 (3x) + 0,739 (3x)

U(0)=0

U(1/3)= 1,2345

U(2/3)= 2,469

U(1) = 3,7036

These are the **numeric solutions**. And they should be close to those exact solutions written in page 5.