EXERCISES 2 - ODE'S

(1) $$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0 \qquad L = 1\ m, \quad g = 9'8\ m/s^2$$

~~At~~ $t = 1s$: $\theta(1) = 0'4\ rad$; $\frac{d\theta}{dt}(1) = 0\ rad/s$

a) BVP in $(0,1)$ using a 2^{nd} order RK to determine θ at $t = 0$, with 2 and 4 time steps.

RK order 2 $\longrightarrow$ $Y_{i+1} = Y_i + \frac{h}{2}[k_1 + k_2]$ with $k_1 = f(t_i, Y_i)$
$k_2 = f(t_i + h, Y_i + h k_1)$

$$\{Y\} = \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} \theta \\ \frac{d\theta}{dt} \end{Bmatrix} \qquad f = \begin{Bmatrix} \frac{d\theta}{dt} \\ \frac{d^2\theta}{dt^2} \end{Bmatrix} = \begin{Bmatrix} y_2 \\ -\frac{g}{L} y_1 \end{Bmatrix} \qquad \alpha = \begin{Bmatrix} 0'4 \\ 0 \end{Bmatrix} = \begin{Bmatrix} y_1(0) \\ y_2(0) \end{Bmatrix}$$

2 TIME STEPS

$i=0 \quad i=1 \quad i=2$

$$h = \frac{0-1}{2} = -\frac{1}{2}$$

$i = 0 \longrightarrow t_0 = 1$

$$Y_0 = \alpha = \begin{Bmatrix} 0'4 \\ 0 \end{Bmatrix}$$

$i = 1 \longrightarrow t_1 = 0'5$

$$k_1 = f(1, Y_0) = \begin{Bmatrix} 0 \\ -\frac{9'8}{1} \cdot 0'4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -3'92 \end{Bmatrix}$$

$$k_2 = f(0'5, Y_0 + h k_1) = \begin{Bmatrix} 1'96 \\ -3'92 \end{Bmatrix}$$

$$Y_1 = Y_0 - \frac{0'5}{2}\left(\begin{Bmatrix} 0 \\ -3'92 \end{Bmatrix} + \begin{Bmatrix} 1'96 \\ -3'92 \end{Bmatrix} \right) = \begin{Bmatrix} -0'09 \\ 1'96 \end{Bmatrix}$$

$i = 2 \longrightarrow \quad t_2 = 0$

$$k_1 = f(0'5, Y_1) = \begin{Bmatrix} 1'96 \\ 0'882 \end{Bmatrix}$$

$$k_2 = f(0, Y_1 + hk_1) = f\left(0, \begin{Bmatrix} -1'07 \\ 1'519 \end{Bmatrix}\right) = \begin{Bmatrix} 1'519 \\ 10'486 \end{Bmatrix}$$

$$Y_2 = \begin{Bmatrix} -0'09 \\ 1'96 \end{Bmatrix} - \frac{0'5}{2}\left(\begin{Bmatrix} 1'96 \\ 0'882 \end{Bmatrix} + \begin{Bmatrix} 1'519 \\ 10'486 \end{Bmatrix}\right) = \begin{Bmatrix} -0'9598 \\ -0'882 \end{Bmatrix}$$

4 TIME STEPS

$i = 0 \quad t_0 = 1$

$Y_0 = \alpha$

$i = 1 \quad t_1 = 0'75$

$$k_1 = f(1, Y_0) = \begin{Bmatrix} 0 \\ -3'92 \end{Bmatrix}$$

$$k_2 = f(0'75, Y_0 + hk_1) = \begin{Bmatrix} 0'98 \\ -3'92 \end{Bmatrix}$$

$$Y_1 = Y_0 + \frac{h}{2}(k_1 + k_2) = \begin{Bmatrix} 0'2775 \\ 0'98 \end{Bmatrix}$$

$i = 2 \quad t_2 = 0'5$

$$k_1 = f(0'75, Y_1) = \begin{Bmatrix} 0'98 \\ -2'7195 \end{Bmatrix}$$

$$k_2 = f(0'5, Y_1 + hk_1) = \begin{Bmatrix} 1'6599 \\ -0'3185 \end{Bmatrix}$$

$$Y_2 = Y_1 + \frac{h}{2}(k_1 + k_2) = \begin{Bmatrix} -0'0525 \\ 1'3598 \end{Bmatrix}$$

$i = 3 \qquad t_3 = 0'25$

$$k_1 = f(0'5, Y_2) = \begin{Bmatrix} 1'3598 \\ 0'5143 \end{Bmatrix}$$

$$k_2 = f(0'25, Y_2 + hk_1) = \begin{Bmatrix} 1'2312 \\ 3'8457 \end{Bmatrix}$$

$$Y_3 = Y_2 + \frac{h}{2}(k_1 + k_2) = \begin{Bmatrix} -0'3763 \\ 0'8147 \end{Bmatrix}$$

$i = 4 \qquad t_4 = 0$

$$k_1 = f(0'25, Y_3) = \begin{Bmatrix} 0'8147 \\ 3'6882 \end{Bmatrix}$$

$$k_2 = f(0, Y_3 + hk_1) = \begin{Bmatrix} -0'1073 \\ 5'6843 \end{Bmatrix}$$

$$Y_4 = Y_3 + \frac{h}{2}(k_1 + k_2) = \begin{Bmatrix} -0'4648 \\ -0'3568 \end{Bmatrix}$$

b) Using the approximations in a), compute an approximation of the relative error in the solution computed with 2 steps.

An approximation of the analytical solution is obtained by solving the problem using RK4 and $n = 200$.

That yields : $Y_{200} = \begin{Bmatrix} -0'4 \\ 0'0137 \end{Bmatrix}$

Then the relative error can be computed as :

$$\text{error} = \left| \frac{Y_{200} - Y_2}{Y_{200}} \right| 100 \cong 140\%$$

c) Propose a time step h to obtain an approximation with a relative error three orders of magnitude smaller.

$$h^* = \left(\frac{tol}{\varepsilon_h}\right)^{\frac{1}{p+1}} \cdot h$$

$$h^* = (10^{-3})^{\frac{1}{2}} \cdot h = \boxed{10^{-3/2} \cdot h}$$

$$h^* = (-0'25)\, 10^{-3/2} = -7'9 \cdot 10^{-3}\ s$$

(2) IVP :
$$\frac{dy}{dx} = y - x^2 + 1 \qquad x \in (0,1)$$
$$y(0) = 1$$

a) Euler with $h = 0'25$

$Y_0 = \alpha$

$Y_{i+1} = Y_i + h \cdot f(x_i, Y_i)$

$i = 0 \to Y_1 = Y_0 + 0'25\ f(0, Y_0)$

$Y_1 = 1 + 0'25 \cdot (1 - 0 + 1) = 1'5$

$i = 1 \to Y_2 = Y_1 + 0'25\ (1'5 - 0'25^2 + 1) = 2'1094$

$i = 2 \to Y_3 = Y_2 + 0'25\ (2'1094 - 0'5^2 + 1) = 2'8242$

$i = 3 \to Y_4 = Y_3 + 0'25\ (2'8242 - 0'75^2 + 1) = 3'6396$

b) Compute the solution using Heun method with step h such that the computational cost is the same as in a).

In Euler, one function evaluation is needed at each step.

In Heun, two function evaluations are needed at each step. Then

$h_H = 2h_E$ so the computational cost (# of f evaluations) is the same.

$i = 0 \to x_0 = 0$

$Y_0 = 1$

$i = 1 \to x_1 = 0'5$

$k_1 = f(0, Y_0) = 2, \quad k_2 = f(0'5, Y_0 + h k_1) = 2'75$

$Y_1 = 2'1875$

$i = 2 \longrightarrow x_2 = 1$

$k_1 = f(0'5, y_1) = 2'9375, \quad k_2 = f(1, y_1 + h k_1) = 3'6563$

$y_2 = 3'8359$

c) Compute the pure interpolation polynomial that fits the results in b)

x	0	0'5	1
$f(x)$	1	2'1875	3'8359

0	1		
		$\frac{2'1875-1}{0'5} = 2'375$	
0'5	2'1875		$\frac{3'2968-2'375}{1} = 0'9218$
		$\frac{3'8359-2'1875}{0'5} = 3'2968$	
1	3'8359		

$$1 + 2'375(x-0) + 0'9218(x-0)(x-0'5) =$$

$$1 + 2'375x + 0'9218x^2 - 0'4609x =$$

$$= 0'9218x^2 + 1'9141x + 1$$

d) Which approximation criterion do you recomend to fit the results in a)? Compute the polynomial approximation and compare with c)

Use splines to define two polynomials of degree two, one using points x_0, x_1 & x_2 and the other using x_2, x_3 & x_4.

$$\left\{\begin{array}{llll} 0 & 1 & & \\ & & 2 & \\ 0'25 & 1'5 & & 0'8752 \\ & & 2'4376 & \\ 0'5 & 2'1094 & & \end{array}\right.$$

$$1 + 2(x-0) + 0'8752\,(x-0)(x-0'25)$$

$$1 + 2x + 0'8752\,x^2 - 0'2188\,x$$

$$0'8752\,x^2 + 1'7812\,x + 1 \qquad x \in [0, 0'5]$$

$$\left\{\begin{array}{llll} 0'5 & 2'1094 & & \\ & & 2'8592 & \\ 0'75 & 2'8242 & & 0'8048 \\ & & 3'2616 & \\ 1 & 3'6396 & & \end{array}\right.$$

$$2'1094 + 2'8592\,(x-0'5) + 0'8048\,(x-0'5)(x-0'75)$$

$$2'1094 + 2'8592\,x - 1'4296 + 0'8048\,x^2 - 0'6036\,x - 0'4024\,x + 0'3018 =$$

$$0'9816 + 1'8532\,x + 0'8048\,x^2 \qquad x \in \,]0'5, 1]$$

To compare the results, the value of y in $x = 0'4$ is computed using both interpolation polynomials

$$Y_{c)} = 0'9218 \cdot (0'4)^2 + 1'9141\,(0'4) + 1 = 1'91313$$

$$Y_{d)} = 0'8752\,(0'4)^2 + 1'7812\,(0'4) + 1 = 1'85251$$

That gives a relative error between both approximations of:

$$\varepsilon = \frac{|y_{d)} - y_{c)}|}{y_{c)}} = 3'16\ \%$$

Using Heun's method with $h = 0'05$ gives a value of $y_H = 1'9594$ at $x = 0'4$.

Computing the error of each interpolation with respect to y_H yields:

$$\varepsilon_{d)} = \frac{|y_{d)} - y_H|}{y_H} \cdot 100 = 5'45\ \%$$

$$\varepsilon_{c)} = \frac{|y_{c)} - y_H|}{y_H} \cdot 100 = 2'36\ \%$$

(3) $\frac{dy}{dx} = f(x,y)$ $\Omega = [0,1]$

$y(0) = 1$, $y(x)$ exact solution

Foward Euler : $Y_{i+1} = Y_i + h \cdot f(x_i, Y_i)$

a) Deduce the truncation error

According to Taylor's expansion

$$y(x+h) = y(x) + h \frac{dy}{dx} + O(h^2)$$

$$y_{i+1} = y_i + h \frac{dy}{dx} \to \frac{dy}{dx} = \frac{y_{i+1} - y_i}{h} - O(h^2)$$

Replacing in the given equation :

$$\frac{y_{i+1} - y_i}{h} = f(x,y) + O(h^2)$$

$$y_{i+1} = y_i + h \cdot f(x,y) + O(h^2)$$

For consistency, the method should verify $\max Z_i(h) \to 0$ when $h \to 0$

$$Z(h^2) = y_{i+1} - y_i - h f(x,y)$$

The difference between y_{i+1} and y_i tends to zero when h tends to zero ($i+1$ & i coincide) so :

$$\lim_{h \to 0} Z(h^2) = \lim_{h \to 0} -h f(x,y) = 0$$

Then the method is consistent.

b) State backward Euler method

$$y_i = y(x_{i+1} - h) = y(x_{i+1}) - h \left.\frac{dy}{dx}\right|_{x_{i+1}} + O(h^2)$$

$$y_{i+1} = y_i + h\, f(x_{i+1}, y_{i+1})$$

c) $\frac{dy}{dx} = -\lambda y$ (λ real, positive). Deduce stability limits

$f(x, y) = -\lambda y$

For forward Euler method: $y_{i+1} = y_i + h(-\lambda \cdot y_i)$

$$y_{i+1} = (1 - h\lambda)\, y_i = G\, y_i$$

The scheme is absolutely stable if $|1 - h \cdot \lambda| < 1$

The stability condition is $\boxed{h \cdot \lambda > 1}$

For backward Euler method: $y_{i+1} = y_i - h\lambda\, y_{i+1}$

$$(1 + \lambda h)\, y_{i+1} = y_i$$

$$G = \frac{1}{1 + h\lambda}$$

The scheme is absolutely stable if $|G| < 1 \rightarrow \frac{1}{1+h\lambda} < 1$

The stability condition is:

$1 + h\lambda > 1 \rightarrow \boxed{h\lambda > 1}$

d) Backward Euler for $\frac{dy}{dx} = -25\, y^{3's}$ $y(0) = 1$

$h = 1/10$, two steps

$$y_1 = y_0 + \frac{1}{10}\, f\left(\frac{1}{10}, y_1\right)$$

Step 1 :

Define a new function to estimate the value of y_1 :

$$g(y_1) = y_0 + h \cdot f\left(\frac{1}{10}; y_1\right) - y_1 = 0$$

$$g(y_1) = 1 + \frac{1}{10} \cdot (-25\, y_1^{3.5}) - y_1 = 1 - 2'5\, y_1^{3.5} - y_1 = 0$$

A initial estimation of y_1 is needed. Consider that $y_1^{3.5} >> y_1^1$,

$$y_1^{3.5} = \frac{1}{2'5} \rightarrow y_1^0 = 0'76966.$$

Use Newton to solve this new problem : $x^{k+1} = x^k - \frac{f(x^k)}{f'(x^k)}$

$$f'(x^k) = g'(y_1) = -2'5 \cdot 3'5\, y_1^{2'5} - 1 = -0'875\, y_1^{2'5} - 1$$

$$y_1^1 = y_1^0 - \frac{-25\,(y_1^0)^{3'5}}{-0'875\,(y_1^0)^{2'5} - 1} = 0'6309$$

$$y_1^2 = y_1^1 - \frac{-25\,(y_1^1)^{3'5}}{-0'875\,(y_1^1)^{2'5} - 1} = 0'5965$$

$$\boxed{y_1 = 0'5965}$$

Step 2

$$g(y_2) = y_1 + h\, f\left(\frac{2}{10}, y_2\right) - y_2 = 0$$

$$g(y_2) = 0'5965 + \frac{1}{10} \cdot -25\, y_2^{3'5} - y_2 = 0$$

Initial value : $y_2^0 = \left(\frac{0'5965}{2'5}\right)^{\frac{1}{3'5}} = 0'6640$

$$y_2^1 = y_2^0 - \frac{-25(y_2^0)^{3'5}}{-0'875(y_2^0)^{2'5} - 1} = 0'5039$$

$$y_2^2 = y_2^1 - \frac{-25(y_2^1)^{3'5}}{-0'875(y_2^1)^{2'5} - 1} = 0'4518$$

$$\boxed{y_2 = 0'4518}$$

e) Forward Euler, 2 steps, $h = \frac{1}{10}$

$$\frac{dy}{dx} = -25\,y^{3.5}, \quad y(0) = 1$$

$$Y_{i+1} = Y_i + h\,f(x_i, Y_i)$$

$$i = 0 \longrightarrow Y_1 = Y_0 + \frac{1}{10}\left(-25 \cdot 1^{3.5}\right) = 1 - 2'5 = -1'5$$

$$i = 1 \longrightarrow Y_2 = -1'5 - 2'5 \cdot (-1'5)^{3.5} = -1'5 + 10'3338\,i \text{ (UNSTABLE)}$$

f) Indicate the maximum stable interval size

Using MATLAB, the maximum stable interval size is $h = \frac{1}{30}$

For h bigger than $\frac{1}{30}$ the solution turns unstable and the method gives complex values for the solution.

Stability condition is $\quad Y_{i+1} = Y_i + h\left(-\lambda Y_i^{3.5}\right)$

$$Y_{i+1} =$$

To compute the stability condition for this function, it has to be linearized first using a Taylor expansion around $y = 1$

$$f(y) = -25\,y^{3.5} = f(1) + f'(1)(y-1)$$

$$f(y) = -25 - 87'5(y-1) = 62'5 - \overbrace{87'5}^{\lambda}\,y$$

The Euler method is stable when $0 \leq \lambda h \leq 2$

For this case, considering $\lambda = 87'5 \longrightarrow h < \frac{2}{\lambda} = \frac{2}{87'5} = 0'023$

The value obtained with Matlab was $h \simeq 0'033$. The difference between values comes from the linearization.

4 $\frac{d^2y}{dx^2} + \omega^2 \cdot y = 0 \qquad \Omega \; [0, 1]$

$y(0) = 0 , \quad \frac{dy}{dx}(0) = \omega$

a) Reduce it to a 1^{st} order system

$$[z] = \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} y \\ \frac{dy}{dx} \end{Bmatrix} \qquad f(x,y) = \begin{Bmatrix} \frac{dy}{dx} \\ \frac{d^2y}{dx^2} \end{Bmatrix} = \begin{Bmatrix} z_2 \\ -\omega^2 z_1 \end{Bmatrix}$$

$$\alpha = \begin{Bmatrix} 0 \\ \omega \end{Bmatrix}$$

b) set $\omega^2 = 3$. $t = 1$, $n = 4$ steps (Foward Euler)

$$y_{i+1} = y_i + h f(x_i, y_i)$$

$i = 0$

$$y_1 = y_0 + h f(x_0, y_0) = \begin{Bmatrix} 0 \\ \omega \end{Bmatrix} + 0'25 \begin{Bmatrix} \omega \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0'25\,\omega \\ \omega \end{Bmatrix} = \begin{Bmatrix} 0'433 \\ 1'7321 \end{Bmatrix}$$

$i = 1$

$$y_2 = y_1 + h f(x_1, y_1) = \begin{Bmatrix} 0'75 \\ 3 \end{Bmatrix} + 0'25 \begin{Bmatrix} 1'7321 \\ -3 \cdot 0'433 \end{Bmatrix} = \begin{Bmatrix} 0'866 \\ 1'4073 \end{Bmatrix}$$

$i = 2$

$$y_3 = \begin{Bmatrix} 0'866 \\ 1'4073 \end{Bmatrix} + 0'25 \begin{Bmatrix} 1'4073 \\ -3 \cdot 0'866 \end{Bmatrix} = \begin{Bmatrix} 1'2178 \\ 0'7578 \end{Bmatrix}$$

$i = 3$

$$y_4 = \begin{Bmatrix} 1'2178 \\ 0'7578 \end{Bmatrix} + 0'25 \cdot \begin{Bmatrix} 0'7578 \\ -3 \cdot 1'2178 \end{Bmatrix} = \begin{Bmatrix} 1'4073 \\ -0'1556 \end{Bmatrix}$$

c) Using matlab code, compute the solution using $n = 8$ steps. Use this solution to estimate the step size required to obtain a numerical solution with 3 significant digits.

Using $n = 8$ in Matlab yields $y(x=1) = \begin{Bmatrix} 1'1902 \\ -0'2798 \end{Bmatrix}$

$$h^* = \left(\frac{tol}{E_n}\right)^{\frac{1}{p+1}} \cdot h = \left(10^{-3}\right)^{1/2} \cdot h$$

$$h^* = 10^{-3/2} \cdot h = 10^{-3/2} \cdot 0'125 = 0'004 \Rightarrow n = 25 \text{ steps}$$

$$Y_{n=25} = \begin{Bmatrix} 1'0484 \\ -0'2902 \end{Bmatrix}$$

$$Y_{n=26} = \begin{Bmatrix} 1'0459 \\ -0'2899 \end{Bmatrix}$$

$$Y_{n=27} = \begin{Bmatrix} 1'0437 \\ -0'2897 \end{Bmatrix}$$

$|Y_{n=25} - Y_{n=26}| = 0'0025 \Rightarrow$ The numerical solution has 3 significant digits (#.##)