NUMERICAL METHODS FOR PDE's - EXERCISES 3

2. $u_t + a u_x = 0 \qquad x \in (0,1), \quad t \geq 0, \quad a > 0$

$$u(x,0) = \sin(2\pi x)$$

$$u(0,t) = u(1,t)$$

a) Propose an implicit FD scheme, 1st order in time & space. Justify the selection of the approximation for the spatial derivative.

$$\left.\frac{\partial u}{\partial t}\right|_i^{n+1} = \frac{u_i^{n+1} - u_i^n}{\Delta t} + O(\Delta t)$$

$$\left.\frac{\partial u}{\partial x}\right|_i^{n+1} = \frac{u_i^{n+1} - u_{i-1}^{n+1}}{\Delta x} + O(\Delta x)$$

Backward finite difference approximation $\rightarrow$ first order

Writing the equation using those expressions yields:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a \frac{u_i^{n+1} - u_{i-1}^{n+1}}{\Delta x} + O(\Delta t, \Delta x) = 0$$

$$u_i^{n+1} - u_i^n + \overbrace{\frac{a \cdot \Delta t}{\Delta x}}^{c} \left(u_i^{n+1} - u_{i-1}^{n+1} \right) = 0 \qquad \text{(neglecting truncation error)}$$

$$(1+c)\, u_i^{n+1} - c\, u_{i-1}^{n+1} = u_i^n$$

b) How are BC treated? Write in detail the system to be solved

Writing it in matrix notation: $A U^{n+1} = I U^n + B$

$$A = \begin{bmatrix} 1+c & 0 & \cdots & 0 \\ -c & 1+c & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & -c & 1+c \end{bmatrix} \qquad B = \begin{bmatrix} c\, u_0^{n+1} \\ 0 \\ \vdots \\ 0 \\ c\, u_{n+1}^{n+1} \end{bmatrix} = \begin{bmatrix} c\, u_0^{n+1} \\ 0 \\ \vdots \\ 0 \\ c \cdot u_0^{n+1} \end{bmatrix}$$

$\hookrightarrow$ periodic BC.

c), d) :

$$A = \begin{bmatrix} \ddots & & 0 \\ \ddots & \ddots & \\ 0 & \ddots & \ddots \end{bmatrix} \longrightarrow$$ Not symmetric, Cholesky decomposition cannot be applied

DIRECT METHOD :

A can be factorized using Doolittle method . $A = LU$

$$A = \begin{bmatrix} \ddots & & \\ \ddots & \ddots & \\ & \ddots & \ddots \end{bmatrix} = \begin{bmatrix} \ddots & & \\ \ddots & \ddots & \\ & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \ddots & \ddots & \\ & \ddots & \ddots \\ & & \ddots \end{bmatrix}$$

↳ $U = A^{(n-1)}$ ↳ unitary diagonal (L)

The solution is computed with two substitutions

1. $Ly = b \rightarrow y$ (FS)
2. $Ux = y \rightarrow x$ (BS)

ITERATIVE METHOD :

The system can be solved using Gauss-Seidel method :

$$u_i^{k+1} = \frac{1}{a_{ii}} \left[b_i - \sum_{j=1}^{i-1} a_{ij} u_j^{k+1} - \sum_{j=i+1}^{n} a_{ij} u_j^{k} \right] \qquad \begin{array}{l} i = 1, \dots, n \\ k = 0, 1, \dots \end{array}$$

4. $u_t = \nu u_{xx} + \sigma u$ in $x \in (0,1)$, $t > 0$, $\underbrace{\nu > 0, \sigma < 0}_{\text{constant}}$

$u(0,t) = 0$, $u_x(1,t) = 0$

$$u(x,0) = \begin{cases} 0 & \text{for } x < 1/4 \\ 4x-1 & \text{for } 1/4 \leq x < 1/2 \\ -4x+3 & \text{for } 1/2 \leq x < 3/4 \\ 0 & \text{for } 3/4 \leq x \end{cases}$$

a) Explicit FD scheme. Detail numerical treatment of B.C.

$$u_t = \left.\frac{\partial u}{\partial t}\right|_i^n \approx \frac{u_i^{n+1} - u_i^n}{\Delta t} + \mathcal{O}(\Delta t)$$

$$u_{xx} = \left.\frac{\partial^2 u}{\partial x^2}\right|_i^n = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \nu \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} + \sigma u_i^n$$

$$u_i^{n+1} - u_i^n = \frac{\Delta t\, \nu}{\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) + \Delta t\, \sigma\, u_i^n$$

$$u_i^{n+1} = \frac{\Delta t\, \nu}{\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) + (\Delta t\, \sigma + 1)\, u_i^n$$

$$u_i^{n+1} = \nu \frac{\Delta t}{\Delta x^2} u_{i+1}^n + \left(1 + \sigma \Delta t - 2\nu \frac{\Delta t}{\Delta x^2}\right) u_i^n + \nu \frac{\Delta t}{\Delta x^2} u_{i-1}^n \quad \text{for } i = 0, \dots, M, \; n \geq 0$$

To take into account the boundary conditions & initial conditions

I.C.: $u_i^0 = u(x,0)$ for $i = 0, \dots, M+1$

B.C. (D): $u_0^{n+1} = 0$ $n \geq 0$

$$\text{BC (N)}: \quad u_{M+1}^{n+1} = \nu \frac{\Delta t}{\Delta x^2} \underbrace{u_{M+2}^n}_{\text{Ghost node}} + \left(1 + \sigma \Delta t - 2\nu \frac{\Delta t}{\Delta x^2}\right) u_{M+1}^n + \nu \frac{\Delta t}{\Delta x^2} u_M^n$$

Approximating the B.C

$$u_x \Big|_{M+1}^{p} = \frac{u_{M+2}^{p} - u_{M}^{p}}{2\Delta x} = 0 \implies u_{M+2}^{p} = u_{M}^{p}$$

Then $$u_{M+1}^{n+1} = 2\frac{\nu \Delta t}{\Delta x^2} u_M^n + \left(1 + \sigma \Delta t - \frac{2\nu\Delta t}{\Delta x^2}\right) u_{M+1}^n$$

Summarizing the scheme will be

$$u_i^{n+1} = \nu\frac{\Delta t}{\Delta x^2} u_{i-1}^n + \left(1 + \sigma\Delta t - 2\nu\frac{\Delta t}{\Delta x^2}\right) u_i^n + \nu\frac{\Delta t}{\Delta x^2} u_{i+1}^n , \quad \text{for } i = 1, M$$

$$u_{M+1}^{n+1} = 2\nu\frac{\Delta t}{\Delta x^2} u_M^n + \left(1 + \sigma\Delta t - 2\nu\frac{\Delta t}{\Delta x^2}\right) u_{M+1}^n$$

$$u_0^{n+1} = 0$$

$u_i^0 = f(x)$ (given in the problem) for $i = 1, \dots, M+1$

b) Schemes obtained for $\sigma = 0$ (diff. eq.)? And for $\nu = 0$ (reaction eq.)?

- For $\sigma = 0$

$$u_i^{n+1} = \nu\frac{\Delta t}{\Delta x^2} u_{i-1}^n + \left(1 - 2\nu\frac{\Delta t}{\Delta x^2}\right) u_i^n + \nu\frac{\Delta t}{\Delta x^2} u_{i+1}^n \quad \text{for } i = 1, \dots M$$

$$u_{M+1}^{n+1} = 2\nu\frac{\Delta t}{\Delta x^2} u_M^n + \left(1 - 2\nu\frac{\Delta t}{\Delta x^2}\right) u_{M+1}^n$$

$$u_0^{n+1} = 0$$

$$u_i^0 = f(x)$$

- For $\nu = 0$

$$u_i^{n+1} = (1 + \sigma\Delta t)\, u_i^n$$

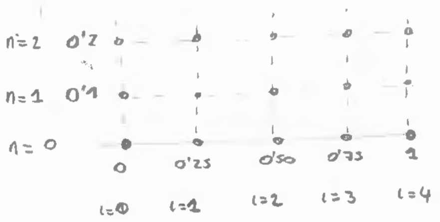
$$u_{M+1}^{n+1} = (1 + \sigma\Delta t)\, u_{M+1}^n$$

$$u_0^{n+1} = 0$$

$$u_i^0 = f(x)$$

c) $\nu = 0.1$, $\sigma = -0.1$. $\Delta x = 0'25$, $\Delta t = 0'1$.

Mesh:



$$u_i^{n+1} = \frac{4}{25} u_{i-1}^n + \frac{67}{100} u^n + \frac{4}{25} u_{i+1}^n \qquad \text{for } i = 1, 2, 3$$

$$u_4^{n+1} = \frac{8}{25} u_3^n + \frac{67}{100} u_4^n \qquad n = 1, 2$$

$$u_0^{n+1} = 0$$

$$u_i^0 = f(x) \qquad \text{for } i = 1, \dots 4$$

- $n = 0 \implies u_0^0 = 0$, $u_1^0 = 0$, $u_2^0 = 1$, $u_3^0 = 0$, $u_4^0 = 0$
- $n = 1 \implies u_0^1 = 0$, $u_1^1 = \frac{4}{25}$, $u_2^1 = \frac{67}{100}$, $u_3^1 = \frac{4}{25}$, $u_4^1 = 0$
- $n = 2 \implies u_0^2 = 0$, $u_1^2 = \frac{134}{625}$, $u_2^2 = 10000$, $u_3^2 = \frac{134}{625}$, $u_4^2 = \frac{32}{625}$

As it can be seen in the results obtained, there is an important diffusion effect in the problem and the solutions grows with time. The scheme is not useful and the results are not realistic.

d) Propose an implicit scheme

To obtain an implicit scheme use backward difference for time. Central difference in space can be used for accuracy.

$$\left.\frac{\partial u}{\partial t}\right|_i^{n+1} = \frac{u_i^{n+1} - u_i^n}{\Delta t} + O(\Delta t)$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_i^{n+1} = \frac{u_{i-1}^{n+1} + u_{i+1}^{n+1} - 2u_i^{n+1}}{\Delta x^2} + O(\Delta x^2)$$

Then the numerical scheme is :

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\nu}{\Delta x^2} \left(u_{i-1}^{n+1} + u_{i+1}^{n+1} - 2u_i \right) + \sigma u_i^{n+1}$$

$$u_i^n = -\nu \frac{\Delta t}{\Delta x^2} u_{i-1}^{n+1} + \left(1 + 2\nu \frac{\Delta t}{\Delta x^2} - \sigma \Delta t\right) u_i^{n+1} - 2\nu \frac{\Delta t}{\Delta x^2} u_{i+1}^{n+1} \qquad \text{for } i = 1, \ldots, M$$

B.C are treated as before so :

$$u_{M+1}^n = -2 \frac{\nu \Delta t}{\Delta x^2} u_M^{n+1} + \left(1 + 2\nu \frac{\Delta t}{\Delta x^2} - \sigma \Delta t\right) u_{M+1}^{n+1}$$

$$u_0^{n+1} = 0$$

$$u_i^0 = f(x)$$

In matrix form $A \cdot U^{n+1} = I \; u^n$

$$A = \begin{bmatrix} 1 + 2\nu\frac{\Delta t}{\Delta x^2} - \sigma \Delta t & -\nu\frac{\Delta t}{\Delta x^2} & 0 & \cdots & 0 \\ -\nu\frac{\Delta t}{\Delta x^2} & 1 + 2\nu\frac{\Delta t}{\Delta x^2} - \sigma\Delta t & -\nu\frac{\Delta t}{\Delta x^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -\nu\frac{\Delta t}{\Delta x^2} \\ 0 & \cdots & & -2\nu\frac{\Delta t}{\Delta x^2} & 1 + 2\nu\frac{\Delta t}{\Delta x^2} - \sigma\Delta t \end{bmatrix}$$

Tridiagonal system $\longrightarrow$ solve using Thomas algorithm