FINITE DIFFERENCES EXERCISES

[2] $\dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial x} = 0 \qquad x \in (0,1) \quad t \geq 0, \; a > 0$

$u(x,0) = \sin(2\pi x)$

periodic boundary condition $u(0,t) = u(1,t)$

a) Implicit scheme $O(\Delta t, \Delta x)$

$$u_t\Big|_i^{n+1} = \frac{u_i^{n+1} - u_i^n}{\Delta t} + O(\Delta t) \qquad \text{(backward in time)}$$

$$u_x\Big|_i^{n+1} = \frac{u_i^{n+1} - u_{i-1}^{n+1}}{\Delta x} + O(\Delta x) \qquad \text{(backward in space)}$$

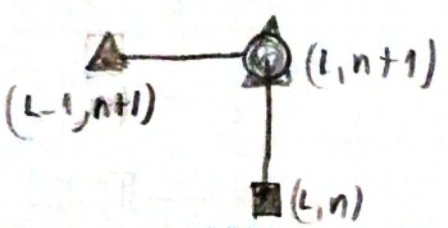
$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\,\frac{u_i^{n+1} - u_{i-1}^{n+1}}{\Delta x} + O(\Delta t, \Delta x) = 0$$

Neglecting truncation errors,

$$U_i^{n+1} - U_i^n + a\frac{\Delta t}{\Delta x}\left(U_i^{n+1} - U_{i-1}^{n+1}\right) = 0$$

$$\boxed{\left(1 + a\frac{\Delta t}{\Delta x}\right)U_i^{n+1} - \frac{a\Delta t}{\Delta x}\,U_{i-1}^{n+1} = U_i^n}$$

For the space approximation, centered differences are not used as they are second order.



b)

for $i=0$

$$\left(1+a\frac{\Delta t}{\Delta x}\right)U_0^{n+1}-\frac{a\Delta t}{\Delta x}U_{-1}^{n+1}=U_0^n$$

but $U_{-1}^{n+1}=U_M^{n+1}$; $U_0^{n+1}=U_{M+1}^{n+1}$ (periodic b.c.)

$$\left(1+\frac{a\Delta t}{\Delta x}\right)U_0^{n+1}-\frac{a\Delta t}{\Delta x}U_M^{n+1}=U_0^n$$

for $i=M$

$$\left(1+a\frac{\Delta t}{\Delta x}\right)U_{M+1}^{n+1}-a\frac{\Delta t}{\Delta x}U_{M-1}^{n+1}=U_M^n$$

for $i=M+1$

$$\left(1+a\frac{\Delta t}{\Delta x}\right)U_{M+1}^{n+1}-\frac{a\Delta t}{\Delta x}U_M=U_{M+1}^n$$

$$\left(1+a\frac{\Delta t}{\Delta x}\right)U_0^{n+1}-a\frac{\Delta t}{\Delta x}U_M^{n+1}=U_0^n$$

The system of equations is $AU^{n+1}=U^n$

$$A=\begin{bmatrix}\left(1+\frac{a\Delta t}{\Delta x}\right) & 0 & \cdots & -a\frac{\Delta t}{\Delta x}\\ -a\frac{\Delta t}{\Delta x} & \ddots & \ddots & \\ & \ddots & \ddots & 0\\ & & -a\frac{\Delta t}{\Delta x} & 1+\frac{a\Delta t}{\Delta x}\end{bmatrix}$$

$$U^{n+1} = (U_0^{n+1}, \text{\_\_\_\_}, U_M^{n+1})^T$$

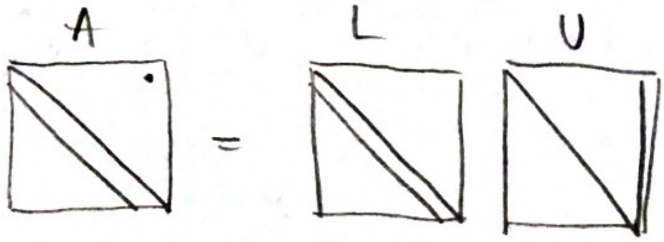
$$U^n = (U_0^n, \text{\_\_\_\_}, U_M^0)^T$$

where the last element on the first row of A appears due to the boundary conditions periodicity. Noto that this equations are for $i = 0, \text{\_\_}, M$ stead of $i = 1, \text{\_\_}, M$. This is then a $M+1 \times M+1$ system of equations of the "implicit upwind scheme".

c) Matrix A is not symmetric then cholesky factorization cannot be used. We can both suggest Doolittle or crout as they are better methods than typical Gauss elimination.

For iterative method, conjugate gradients cannot be used. Then, Jacobi and Gauss-seidel methods can be used and Gauss-seidel is proposed as it usually converges faster.

d)



$\boxed{4}$ $u_t = \nu u_{xx} + \sigma u \qquad x \in [0,1] \quad t > 0$

$u(0,t) = 0, \quad u_x(1,t) = 0 \; h(t)$

$$u(x,0) = \begin{cases} 0 & x < 1/4 \\ 4x-1 & 1/4 \le x < 1/2 \\ -4x+3 & 1/2 \le x < 3/4 \\ 0 & 3/4 \le x \end{cases}$$

$\nu > 0$ diffusion coeff. $\quad \sigma < 0$ reaction coeff.

a) Explicit scheme (FTCS)

$$u_t \Big|_i^n = \frac{u_i^{n+1} - u_i^n}{\Delta t} + O(\Delta t) \quad (FD)$$

$$u_{xx} \Big|_i^n = \frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{\Delta x^2} + O(\Delta x^2) \quad (CD)$$

The scheme is (neglecting truncation errors)

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = \nu \frac{U_{i-1}^n - 2U_i^n + U_{i+1}^n}{\Delta x^2} + \sigma U_i^n$$

$$U_i^{n+1} = U_i^n + \sigma \Delta t \, U_i^n + \underbrace{\frac{\nu \Delta t}{\Delta x^2}}_{r}(U_{i-1}^n - 2U_i^n + U_{i+1}^n)$$

$$U_i^{n+1} = r U_{i-1}^n + (1 - 2r + \sigma \Delta t) U_i^n + r U_{i+1}^n$$

IC: $U_i^0 = u(x_i, 0) \qquad \forall \; i = 0, \ldots, M+1$

BC: Dirichlet $\quad U_0^{n+1} = 0 \qquad n \ge 0$

Neumann $\Rightarrow$ fictitious nodes

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scheme @ $i = M+1$

$$U_{M+1}^{n+1} = r\,U_M^n + (1-2r+\sigma\Delta t)\,U_{M+1}^n + r\,U_{M+2}^n$$

$$\left.\frac{\partial u}{\partial x}\right|_{M+1}^{n} = \frac{u_{M+2}^n - u_M^n}{2\Delta x} + O(\Delta x^2) = h^n$$

$$\Rightarrow U_{M+2}^n = U_M^n + \cancel{2\Delta x\, h^n} \Rightarrow \boxed{U_{M+2}^n = U_M^n}$$

$$U_{M+1}^{n+1} = r\,U_M^n + (1-2r+\sigma\Delta t)\,U_{M+1}^n + r\,U_M^n =$$

$$= 2r\,U_M^n + (1-2r+\sigma\Delta t)\,U_{M+1}^n$$

Finally,

$$U_i^{n+1} = r\,U_{i-1}^n + (1-2r+\sigma\Delta t)\,U_i^n + r\,U_{i+1}^n \qquad i = 1, \ldots, M \quad n \geq 0$$

$$U_{M+1}^{n+1} = 2r\,U_M^n + (1-2r+\sigma\Delta t)\,U_{M+1}^n \qquad n \geq 0$$

$$U_0^{n+1} = 0$$

$$U_i^0 = f_i \qquad \forall\, i = 0, \ldots, M+1$$

b) • If $\nu = 0 \Rightarrow r = 0$ (reaction equation)

$$U_i^{n+1} = (1+\sigma\Delta t)\, U_i^n \qquad i = 1, \dots, M$$

$$U_{M+1}^{n+1} = (1+\sigma\Delta t)\, U_{M+1}^n \qquad n \geq 0$$

$$U_0^{n+1} = 0 \quad n \geq 0$$

$$U_i^0 = f_i \quad \forall\, i = 0, \dots, M+1$$

• If $\sigma = 0$ (diffusion equation)

$$U_i^{n+1} = r\, U_{i-1}^n + (1-2r)\, U_i^n + r\, U_{i+1}^n \quad i = 1, \dots, M \; n \geq 0$$

$$U_{M+1}^{n+1} = 2r\, U_M^n + (1-2r)\, U_{M+1}^n \quad n \geq 0$$

$$U_0^{n+1} = 0 \qquad n \geq 0$$

$$U_i^0 = f_i \qquad i = 1, \dots, M+1$$

d) Implicit scheme (BCTS)

$$u_t\Big|_i^{n+1} = \frac{u_i^{n+1} - u_i^n}{\Delta t} + O(\Delta t) \quad \text{(BD)}$$

$$u_{xx}\Big|_i^{n+1} = \frac{u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}}{\Delta x^2} + O(\Delta x^2) \quad \text{(CD)}$$

Neglecting truncation errors,

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = \nu \frac{U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}}{\Delta x^2} + \sigma U_i^{n+1}$$

$$U_i^{n+1} - U_i^n = \underbrace{\frac{\nu \Delta t}{\Delta x^2}}_{r} \left(U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}\right) + \sigma \Delta t\, U_i^{n+1}$$

$$-r U_{i-1}^{n+1} + U_i^{n+1} + 2r U_i^{n+1} - \sigma \Delta t\, U_i^{n+1} - r U_{i+1}^{n+1} = U_i^n$$

$$-r U_{i-1}^{n+1} + (1 + 2r - \sigma \Delta t) U_i^{n+1} - r U_{i+1}^{n+1} = U_i^n$$

Neuman b.c

for $i = M+1$

$$-r U_M^{n+1} + (1 + 2r - \sigma \Delta t) U_{M+1}^{n+1} - r U_{M+2}^{n+1} = U_{M+1}^n$$

$$\frac{\partial u}{\partial x}\Big|_{M+1}^{p} = \frac{u_{M+2}^p - u_M^p}{2\Delta x} + O(\Delta x^2) = h^p = 0$$

$$U_{M+2}^p = U_M^p$$

$$-rU_M^{n+1} + (1+2r-\sigma\Delta t)U_{M+1}^{n+1} - rU_M^{n+1} = U_{M+1}^{n}$$

$$-2rU_M^{n+1} + (1+2r-\sigma\Delta t)U_{M+1}^{n+1} = U_{M+1}^{n}$$

The final scheme is

$$-rU_{i-1}^{n+1} + (1+2r-\sigma\Delta t)U_i^{n+1} - rU_{i+1}^{n+1} = U_i^n \qquad \begin{matrix} n \geq 0 \\ i = 1, \ldots, M \end{matrix}$$

$$-2rU_M^{n+1} + (1+2r-\sigma\Delta t)U_{M+1}^{n+1} = U_{M+1}^{n} \qquad n \geq 0$$

$$U_0^{n+1} = 0 \qquad n \geq 0$$

$$U_i^0 = f_i \qquad i = 0, \ldots, M+1$$

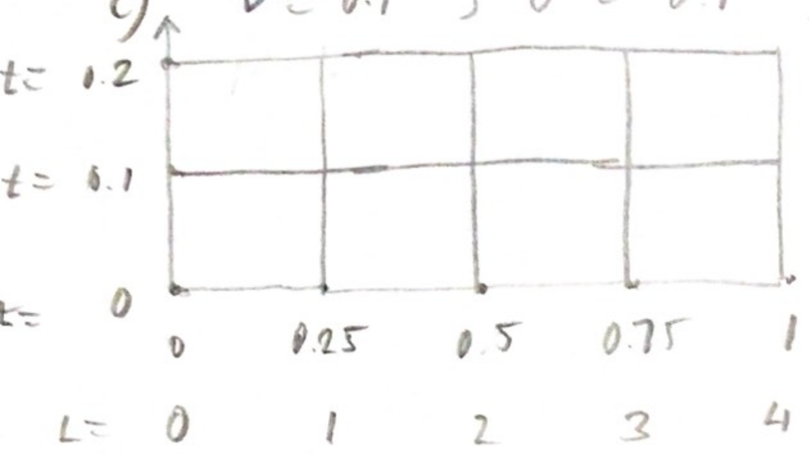
This can be rewritten in matrix form $AU^{n+1} = U^n$

$$A = \begin{bmatrix} 1+2r-\sigma\Delta t & -r & & \\ -r & 1+2r-\sigma\Delta t & \ddots & \\ & \ddots & \ddots & -r \\ & & -2r & 1+2r-\sigma\Delta t \end{bmatrix}$$

Neuman b.c.

This is a tridiagonal system so we can use Thomas algorithm to solve it efficiently as the coefficients are constant for each step.

c) $\nu = 0.1$, $\sigma = -0.1$, $\Delta x = 0.25$ $\Delta t = 0.1$



We can solve on Matlab and it gives

$U_0^0 = 0$; $U_1^0 = 0$; $U_2^0 = 1$; $U_3^0 = 0$; $U_4^0 = 0$

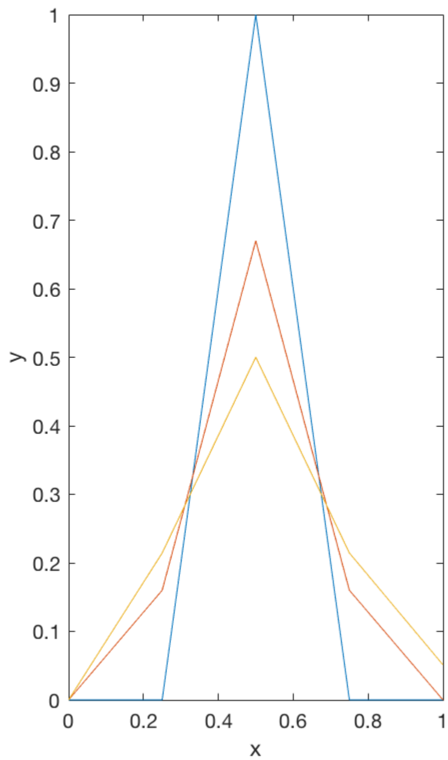
$U_0^1 = 0$; $U_1^1 = 0.1600$; $U_2^1 = 0.6700$ $U_3^1 = 0.1600$ $U_4^1 = 0$

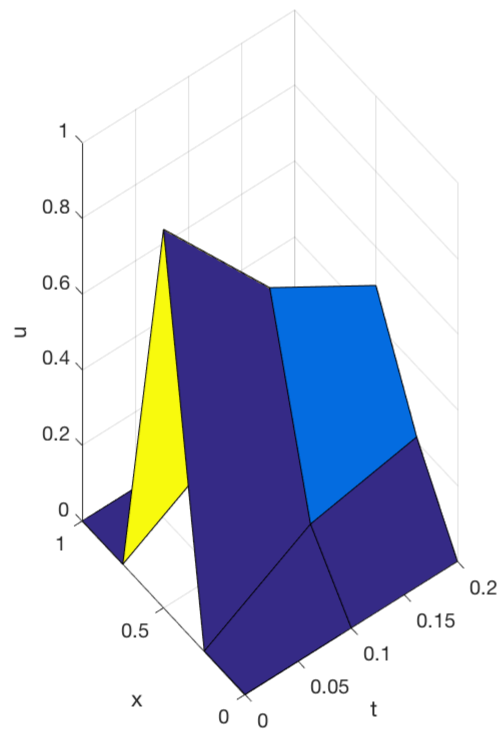
$U_0^2 = 0$; $U_1^2 = 0.2144$; $U_2^2 = 0.5001$ $U_3^2 = 0.2144$ $U_4^2 = 0.0512$

We have a positive diffusion coefficient, meaning that the pollutant is diffusing over time and that is why the profile of the solution is becoming wider.

As the reaction coefficient is negative, this means that some contaminant is being "taking away" from the system.

```
clear all
clc
nu = 0.1; sigma = -0.1; deltax = 0.25; deltat = 0.1;
r = nu*deltat/deltax^2;
a = 0 ; b = 1; m = (b-a)/deltax;
tfinal = 0.2 ; npast = tfinal/deltat ;
%%%%% Initial condition %%%%
x = a:deltax:b;
f = InitialCondition(x);
g=0; % Dirichlet bc
h=0; % Neumann bc
U = zeros(m+1,npast+1);
U(1,:)=g;
U(:,1) = f;
for j = 2 : npast+1
    for i = 2 : m+1
        if i == m +1
            U(i,j) = (1+sigma*deltat - 2*r)*U(i,j-1) +
 2*r*U(i-1,j-1) ;
            break
        end
        U(i,j) = r*U(i-1,j-1) + (1+sigma*deltat - 2*r)*U(i,j-1) +
 r*U(i+1,j-1) ;
    end
end
t = 0:deltat:tfinal;
subplot(1,2,2)
surf(t,x,U), axis([0 deltat*npast a b 0 1])
xlabel('t'), ylabel('x'), zlabel('u') ;
subplot(1,2,1)
plot(x,U);
xlabel('x'), ylabel('y')
function f = InitialCondition(x)
f1= 0 ; f2 = 4*x-1 ; f3 = -4*x + 3 ; f4 = 0;
f =
 f1.*(x<1/4)+f2.*((x>=1/4)&(x<0.5))+f3.*((x>=1/2)&(x<0.75))+f4.*(x>=0.75);
end
```





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