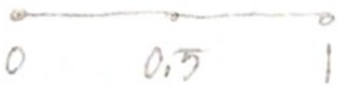
Exercises ODE's

$\boxed{1}$ $\dfrac{d^2\theta}{dt^2} + \dfrac{g}{L}\theta = 0$; $L = 1\,m$, $g = 9.8\ m/s^2$

@ $t = 1\,s \rightarrow \theta(1) = 0.4$ rad , $\dfrac{d\theta}{dt}(1) = 0$ rad/s

a) 2^{nd} order Runge - Kutta with 2 & 4 steps

2 steps :



$h = \dfrac{b-a}{2} = \dfrac{0-1}{2} = -1/2$

$\underline{y} = \begin{Bmatrix} \theta \\ \dfrac{d\theta}{dt} \end{Bmatrix} = \begin{Bmatrix} y_{(1)} \\ y_{(2)} \end{Bmatrix}$; $\underline{f} = \dfrac{dy}{dt} = \begin{Bmatrix} d\theta/dt \\ d\theta^2/dt^2 \end{Bmatrix} = \begin{Bmatrix} y_{(2)} \\ -\frac{g}{L} y_{(1)} \end{Bmatrix}$

$\underline{\alpha} = \begin{Bmatrix} \theta(1) \\ d\theta/dt\,(1) \end{Bmatrix} = \begin{Bmatrix} 0.4 \\ 0 \end{Bmatrix} = \underline{y}_0$

- $y_1 = y_0 + \dfrac{h}{2}[K_1 + K_2]$

$K_1 = f(x_i, Y_i) = \begin{Bmatrix} 0 \\ -\frac{9.8}{1}(0.4) \end{Bmatrix} = \begin{Bmatrix} 0 \\ -3.92 \end{Bmatrix}$

$K_2 = f(x_i + h, Y_i + h K_1) = f\left(x_i + h, \begin{Bmatrix} 0.4 \\ 0 \end{Bmatrix} + h \begin{Bmatrix} 0 \\ -3.92 \end{Bmatrix}\right) =$

$= f\left(x_i + h, \begin{Bmatrix} 0.4 \\ 1.96 \end{Bmatrix}\right) = \begin{Bmatrix} 1.96 \\ -3.92 \end{Bmatrix}$

$y_1 = \begin{Bmatrix} 0.4 \\ 0 \end{Bmatrix} - 1/4 \begin{Bmatrix} 1.96 \\ -7.84 \end{Bmatrix} = \begin{Bmatrix} -0.09 \\ 1.96 \end{Bmatrix}$

$\cdot\ y_2 = y_1 + \frac{h}{2}[K_1 + K_2]$

$$K_1 = f(x_i, Y_i) = f\left(0.5, \begin{Bmatrix} -0.09 \\ 1.96 \end{Bmatrix}\right) = \begin{Bmatrix} 1.96 \\ 0.882 \end{Bmatrix}$$

$$K_2 = f(x_i + h, Y_i + hK_1) = f\left(0, \begin{Bmatrix} -0.09 \\ 1.96 \end{Bmatrix} - 1/2 \begin{Bmatrix} 1.96 \\ 0.882 \end{Bmatrix}\right)$$

$$= f\left(0, \begin{Bmatrix} -1.07 \\ 1.519 \end{Bmatrix}\right) = \begin{Bmatrix} 1.519 \\ (-9.8)(-1.07) \end{Bmatrix} = \begin{Bmatrix} 1.519 \\ 10.486 \end{Bmatrix}$$

$$y_2 = \begin{Bmatrix} -0.09 \\ 1.96 \end{Bmatrix} - 1/4 \left[\begin{Bmatrix} 1.519 \\ 10.486 \end{Bmatrix} + \begin{Bmatrix} 1.96 \\ 0.882 \end{Bmatrix} \right] =$$

$$= \begin{Bmatrix} -0.9598 \\ -0.8820 \end{Bmatrix} \Rightarrow \theta(0) = -0.9598 \ ; \ \frac{d\theta}{dt}(0) = -0.8820$$

<u>4 steps</u>

0 0.25 0.5 0.75 1

$$h = \frac{b-a}{4} = \frac{0-1}{4} = 0.25$$

$\cdot\ y_1 = y_0 + \frac{h}{2}[K_1 + K_2]$

$$K_1 = f(x_i, Y_i) = \begin{Bmatrix} 0 \\ -3.92 \end{Bmatrix}$$

$$K_2 = f(x_i + h, Y_i + hK_1) = f\left(x_i + h, \begin{Bmatrix} 0.4 \\ 0 \end{Bmatrix} - 0.25 \begin{Bmatrix} 0 \\ -3.92 \end{Bmatrix}\right)$$

$$= f\left(x_i + h, \begin{Bmatrix} 0.4 \\ 0.98 \end{Bmatrix}\right) = \begin{Bmatrix} 0.98 \\ -3.92 \end{Bmatrix}$$

$$y_1 = \begin{Bmatrix} 0.4 \\ 0 \end{Bmatrix} - \frac{0.25}{2}\left[\begin{Bmatrix} 0 \\ -3.92 \end{Bmatrix} + \begin{Bmatrix} 0.98 \\ -3.92 \end{Bmatrix} \right] = \begin{Bmatrix} 0.2775 \\ 0.98 \end{Bmatrix}$$

$\cdot\ y_2 = y_1 + h/2\ [K_1 + K_2]$

$K_1 = f(x_i, Y_i) = \begin{Bmatrix} 0.98 \\ -2.7195 \end{Bmatrix}$

$K_2 = f(x_i + h, Y_i + hK_1) = f(x_i + h, \begin{Bmatrix} 0.2775 \\ 0.98 \end{Bmatrix} - 0.25 \begin{Bmatrix} 0.98 \\ -2.7195 \end{Bmatrix}) =$

$= f(x_i + h, \begin{Bmatrix} 0.0325 \\ 1.6598 \end{Bmatrix}) = \begin{Bmatrix} 1.6598 \\ -0.3185 \end{Bmatrix}$

$y_2 = \begin{Bmatrix} 0.2775 \\ 0.98 \end{Bmatrix} - \frac{0.25}{2} \left[\begin{Bmatrix} 0.98 \\ -2.7195 \end{Bmatrix} + \begin{Bmatrix} 1.6598 \\ -0.3185 \end{Bmatrix} \right] =$

$= \begin{Bmatrix} -0.0525 \\ 1.3598 \end{Bmatrix}$

$\cdot\ y_3 = y_2 + h/2\ [K_1 + K_2]$

$K_1 = f(x_i, Y_i) = \begin{Bmatrix} 1.3598 \\ (-9.8)(-0.0525) \end{Bmatrix} = \begin{Bmatrix} 1.3598 \\ 0.5145 \end{Bmatrix}$

$K_2 = f(x_i + h, Y_i + hK_1) = f(x_i + h, \begin{Bmatrix} -0.0525 \\ 1.3598 \end{Bmatrix} + h \begin{Bmatrix} 1.3598 \\ 0.5145 \end{Bmatrix}) =$

$= f(0.25, \begin{Bmatrix} -0.3925 \\ 1.2312 \end{Bmatrix}) = \begin{Bmatrix} 1.2312 \\ (-9.8)(-0.3925) \end{Bmatrix} = \begin{Bmatrix} 1.2312 \\ 3.8465 \end{Bmatrix}$

$y_3 = \begin{Bmatrix} -0.0525 \\ 1.3598 \end{Bmatrix} - \frac{0.25}{2} \left[\begin{Bmatrix} 1.3598 \\ 0.5145 \end{Bmatrix} + \begin{Bmatrix} 1.2312 \\ 3.8465 \end{Bmatrix} \right] =$

$= \begin{Bmatrix} -0.3763 \\ 0.8147 \end{Bmatrix}$

• $y_4 = y_3 + \frac{h}{2}[k_1 + k_2]$

$$k_1 = f(x_i, y_i) = \begin{Bmatrix} 0.8147 \\ 3.6877 \end{Bmatrix}$$

$$k_2 = f\left(0, \begin{Bmatrix} -0.3763 \\ 0.8147 \end{Bmatrix} - 0.25 \begin{Bmatrix} 0.8147 \\ 3.6877 \end{Bmatrix}\right) =$$

$$= f\left(0, \begin{Bmatrix} -0.5799 \\ -0.1072 \end{Bmatrix}\right) = \begin{Bmatrix} -0.1072 \\ (-9.8)(-0.57999) \end{Bmatrix} = \begin{Bmatrix} -0.1072 \\ 5.6830 \end{Bmatrix}$$

$$y_4 = \begin{Bmatrix} -0.3763 \\ 0.8147 \end{Bmatrix} - \frac{0.25}{2}\left[\begin{Bmatrix} -0.1072 \\ 5.6830 \end{Bmatrix} + \begin{Bmatrix} 0.8147 \\ 3.6877 \end{Bmatrix}\right] =$$

$$= \begin{Bmatrix} -0.4648 \\ -0.3568 \end{Bmatrix} \rightarrow \theta(0) = -0.4648 \; ; \; \frac{d\theta}{dt}(0) = -0.3568$$

b) Compute approximation of relative error for solution with 2 steps

Using Matlab we compute a better solution using a high value of steps giving $\theta(0) = -0.4$

$$E_r = \left| \frac{-0.9598 - (-0.4)}{-0.4} \right| = 1.39$$

c)

$$h^* = \left(\frac{tol}{E_h}\right)^{1/(p+1)} \cdot h \qquad ; \; tol = E_h \times 10^{-3}$$

order of the method

$$h^* = \left(\frac{\cancel{E_h} \cdot 10^{-3}}{\cancel{E_h}}\right)^{1/2} \cdot h = \sqrt{10^{-3}} \cdot h$$

For 2 steps $\rightarrow h = 0.5 \rightarrow h^* = 0.016$

$\boxed{2}$ $\frac{dy}{dx} = y - x^2 + 1 \qquad x \in (0,1) \qquad y(0) = 1$

a) Euler method $h = 0.25$

0 0.25 0.5 0.75 1

$Y_0 = \alpha$

$Y_{i+1} = Y_i + h f(x_i, Y_i)$

$y_0 = 1$

$y_1 = y_0 + h f(x_0, y_0) = 1 + 0.25 \cdot f(0,1) =$

$= 1 + 0.25(1 - 0^2 + 1) = 1.5$

$y_2 = y_1 + h f(x_1, y_1) = 1.5 + 0.25 f(0.25, 1.5) =$

$= 1.5 + 0.25(1.5 - 0.25^2 + 1) = 2.1093$

$y_3 = y_2 + h f(x_2, y_2) = 2.1093 + 0.25 \cdot f(0.5, 2.1093)$

$= 2.8242$

$y_4 = y_2 + h f(x_3, y_3) = 2.8242 + 0.25 f(0.75, 2.8242) =$

$= 3.6396$

b) Heun method (RK order 2) $\Rightarrow$ 2 function evaluation per step $\Rightarrow$ same computational cost

$h = 0.5$

$y_1 = y_0 + \frac{h}{2}[K_1 + K_2]$

$K_1 = f(x_0, y_0) = f(0,1) = 1 - 0^2 + 1 = 2$

$k_2 = f(x_0 + h, y_0 + hk_1) = f(1.5, 2) = 2.75$

$y_1 = 1 + 0.25 [2 + 2.5] = 2.1875$

- $y_2 = y_1 + h/2 \, [k_1 + k_2]$

$k_1 = f(x_1, y_1) = f(0.5, 2.1875) = 2.9375$

$k_2 = f(x_1 + h, y_1 + hk_1) = f(1, 3.6563) =$

$= 3.6563 - 1^2 + 1 = 3.6563$

$y_2 = 2.1875 + 0.25 [2.9375 + 3.6563] = 3.8359$

c) $p(x) = ax^2 + bx + c$

x	0	0.5	1
$p(x)$	1	2.1875	3.8359

$p(0) = 1 \rightarrow 1 = c$

$p(0.5) = 2.1875 \rightarrow 2.1875 = a(0.5)^2 + b(0.5) + 1$

$p(1) = 3.8359 \rightarrow 3.8359 = a + b + 1$

$b = 3.8359 - a - 1$

$2.1875 = a(0.5)^2 + (3.8359 - a - 1)(0.5) + 1$

$2.1875 - 1 + 0.5 - (3.8359)(0.5) = a((0.5)^2 - 0.5)$

$a = 0.9218 \rightarrow b = 1.9141$

$p(x) = 0.9218 x^2 + 1.9141 x + 1$

3 $\frac{dy}{dx} = f(x,y)$ $x \in (0,1)$ $y(0)=1$

Forward Euler method $Y_{i+1} = Y_i + hf(x_i, Y_i)$

$x_{i+1} = x_i + h$

a)

$$y_{i+1} = y_i + h\frac{dy}{dx}(x_i) + O(h^2)$$ Taylor expansion

$$\frac{y_{i+1}-y_i}{h} = \frac{dy}{dx}(x_i) + O(h) \rightarrow \frac{dy}{dx}(x_i) = \frac{y_{i+1}-y_i}{h} - O(h)$$

$\tau_i(h) = O(h) \rightarrow$ truncation error

The error is proportional to h, so when $h \to 0$, $\tau_i(h) \to 0$ $\Rightarrow$ the method is consistent

$$\max_{0 \le i \le m} \tau_i(h) \to 0 \quad \text{when} \quad h \to 0$$

b) Backward Euler $\rightarrow$ backward approximation of the derivative

$$y_i = y_{i+1} - h\frac{dy}{dx}(x_{i+1}) + O(h^2)$$

$$\frac{dy}{dx}(x_{i+1}) = \frac{y_{i+1}-y_i}{h} + \tau_i(h)$$

being $\tau_i(h) = O(h)$ the truncation error

Replacing this in the ODE

$y_{i+1} = y_i + h f(x_{i+1}, y_{i+1}) + h \tau_i(h)$ and

neglecting the truncation error, it gives

$$Y_{i+1} = Y_i + h f(x_{i+1}, Y_{i+1})$$

An equation (or system), needs to be solved in order to compute Y_{i+1} from Y_i → this is an implicit method. Thus

$$Y_{i+1} - Y_i - h f(x_{i+1}, Y_{i+1}) = 0$$

would need to be solved to compute Y_{i+1}

c) $\frac{dy}{dx} = f(x, y) = -\lambda y$

<u>Euler method</u> → $Y_{i+1} = Y_i + h f(x_i, Y_i)$

$$Y_{i+1} = Y_i + h(-\lambda Y_i) = Y_i - h\lambda Y_i = G Y_i$$

being $G = 1 - h\lambda$

The scheme is stable if $|G| < 1 \iff |1 - h\lambda| < 1$

$$-1 < 1 - h\lambda < 1 \quad ; \quad -2 < -h\lambda < 0$$

$-2 < -h\lambda \rightarrow 2 > h\lambda \rightarrow h\lambda < 2$

$-h\lambda < 0 \rightarrow h\lambda > 0$

$$\boxed{0 < h\lambda < 2}$$

Backward Euler $Y_{i+1} = Y_i + h(-\lambda) Y_{i+1}$

$Y_{i+1} = Y_i - h\lambda Y_{i+1}$; $(1 + h\lambda) Y_{i+1} = Y_i$

Calling $G = \dfrac{1}{1 + h\lambda}$ $\rightarrow$ $Y_{i+1} = G Y_i$

In order to be stable $\rightarrow$ $|G| < 1$ $\rightarrow$ $|1 + h\lambda| > 1$

$1 + h\lambda > 1$; $h\lambda > 0$

$-1 - h\lambda > 1$; $-h\lambda > 2$; $h\lambda < -2$

Stability condition $\forall \lambda \in \mathbb{R} \rightarrow h\lambda < -2$ or $h\lambda > 0$

d) $\frac{dy}{dx} = -25y^{3.5}$, $y(0)=1$: 2 steps $\Rightarrow n=2$

$h = 1/10$

Backward Euler → use 2 Newton iterations

$$\begin{cases} Y_0 = \alpha \\ Y_{i+1} = Y_i + hf(x_{i+1}, Y_{i+1}) \quad i=0, \text{—}, m-1 \end{cases}$$

$\cdot\ Y_1 = Y_0 + hf(1/10, Y_1) \rightarrow Y_0 + hf(1/10, Y_1) - Y_1 = 0 = g(Y_1)$

$1 + 1/10\, f(1/10, Y_1) - Y_1 = 1 + \frac{1}{10}(-25\, Y_1^{3.5}) - Y_1 = 0$

$Y_1^{3.5} >> Y_1 \rightarrow 1 + \frac{1}{10}(-25\, Y_1^{3.5}) = 0 \rightarrow Y_1^0 = 0.7697$
(first guess)

$g'(Y_1) = -\frac{25}{10}(3.5)\, Y_1^{2.5} - 1 \rightarrow Y_1' = Y_1^0 - \frac{g(Y_1^0)}{g'(Y_1^0)} =$

$$= 0.7697 - \frac{[1 + 0.1(-25 \cdot (0.7697)^{2.5}) - 0.7697]}{[(-2.5)(3.5)(0.7697)^{2.5} - 1]} = 0.6309$$

$$Y_1^2 = Y_1' - \frac{g(Y_1')}{g'(Y_1')} = 0.6309 - \frac{[1 + 0.1(-25\,(0.6309)^{3.5}) - 0.6309]}{[(-2.5)(3.5)(0.6309)^{2.5} - 1]} =$$

$= 0.5965 \rightarrow \boxed{Y_1 = 0.5965}$

$Y_2 = Y_1 + h f(2/10, Y_2)$

$g(Y_2) = Y_1 + h f(2/10, Y_2) - Y_2 = 0$

$g(Y_2) = 0.5965 + \frac{1}{10}(-25\, Y_2^{3.5}) - Y_2 = 0$

$Y_2^0 \Rightarrow 0.5965 + \frac{1}{10}(-25\, Y_2^{3.5}) = 0 \rightarrow Y_2^0 = 0.6640$

$$Y_2^1 = Y_2^0 - \frac{g(Y_2^0)}{g'(Y_2^0)} = 0.6640 - \frac{0.5965 + 0.1(-25 \times (0.6640)^{3.5}) - 0.6640}{(-2.5)(3.5)(0.6640)^{2.5} - 1} =$$

$= 0.5039$

$$Y_2^2 = Y_2^1 - \frac{g(Y_2^1)}{g'(Y_2^1)} = 0.5039 - \frac{0.5039 + 0.1(-25 \times 0.5039^{3.5}) - 0.5939}{(-2.5)(3.5)(0.5039)^{2.5} - 1} =$$

$= 0.4518 \rightarrow Y_2 = 0.4518$

e) Forward Euler $\rightarrow Y_{i+1} = Y_i + h f(x_i, Y_i)$

$Y_0 = 1$

$Y_1 = Y_0 + h f(x_0, Y_0) = 1 + \frac{1}{10} f(0,1) =$

$= 1 + \frac{1}{10}(-25 \cdot 1^{3.5}) = -1.5$

$Y_2 = Y_1 + h f(x_1, Y_1) = -1.5 + \frac{1}{10} f(\frac{2}{10}, -1.5) =$

$= -1.5 + \frac{1}{10}(-25 \cdot (-1.5)^{3.5}) =$ complex number

The problem is that $h = 1/10$ is not a stable interval size for this problem

f) $$y(x) = \left(\frac{125x+2}{2}\right)^{-2/5}$$

With the Matlab code we see that

$h = 1/10 \rightarrow$ really unstable

$h = 1/15 \rightarrow$ unstable

$h = 1/30 \rightarrow$ OK! $\rightarrow$ maximum stable interval

$h = 1/45 \rightarrow$ OK!

$h = 1/90 \rightarrow$ OK!

In class we studied the case $f(x,y) = \lambda y$.
We can try to linearize the function and then compare

Taylor series: $f(y) = f(1) + \frac{dy}{dx}(1)(y-1)$

with $f(y) = \frac{dy}{dx} = -25\, y^{3.5}$; $\frac{dy}{dx}(1) = (-25)(3.5)(1)^{2.5} = -87.5$

$f(1) = -25(1)^{3.5} = -25$

$f(y) = -25 - 87.5(y-1) = 62.5 - 87.5y = a - \lambda y$

$\simeq -\lambda y$

from part c) we obtained that Euler method is stable for $0 < h\lambda < 2$ for a function of the form $dy/dx = -\lambda y$

$0 < h < \frac{2}{\lambda}$, $h < \frac{2}{87.5} = 0.0229$

Thus

$1/10 > 0.0229 \rightarrow$ not stable interval size

$1/15 > 0.0229 \rightarrow$ " " "

$1/30 > 0.0229 \rightarrow$ " " "

$1/45$, $1/90 < 0.0229 \rightarrow$ stable interval size

We now obtain that $1/30$ is not a stable interval but Matlab showed that it actually is. Of course, we'd introduced an error in the computation as we linearized a function which is not linear.