NUMERICAL METHODS FOR PDE'S

BASICS

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$\boxed{1}$ $f(x) := x^3 + 2x^2 + 10x - 20 = 0$

Find unique real root with 4 iterations of Newton's method

$x_0 = \sqrt[3]{20}$

Solution:

Newton's method $\Rightarrow$ $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$

$f'(x) = 3x^2 + 4x + 10$

$$\left.\begin{aligned} f(\sqrt[3]{20}) &= (\sqrt[3]{20})^3 + 2(\sqrt[3]{20})^2 + 10(\sqrt[3]{20}) - 20 = 41.8803 \\ f'(\sqrt[3]{20}) &= 3(\sqrt[3]{20})^2 + 4(\sqrt[3]{20}) + 10 = 42.9619 \end{aligned}\right\} \Rightarrow$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \sqrt[3]{20} - \frac{41.8803}{42.9619} = 1.7396$$

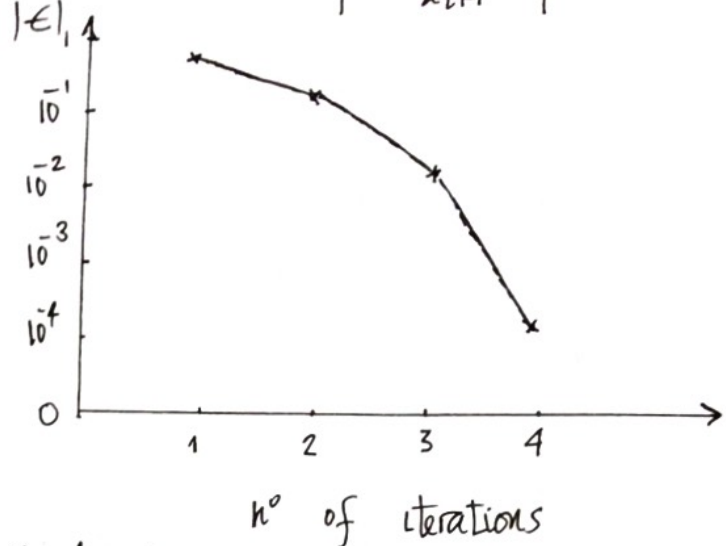
We can construct a table with the iterations

i	x_i	$f(x_i)$	$f'(x_i)$	x_{i+1}	
0	$\sqrt[3]{20}$	41.8803	42.9619	1.7396	
1	1.7396	8.7126	26.0370	1.4049	
2	1.4049	0.7694	21.5403	1.3692	
3	1.3692	7.8655×10^{-3}	21.1009	1.3688	$\approx$ exact solution

As we can see, Newton's method is a rapid procedure if a reasonable initial approximation is given.

Find the relative approximate error $|\epsilon| = \left| \frac{x_{i+1} - x_i}{x_{i+1}} \right|$

Iteration	Error $\|\epsilon\|$
1	0.5604
2	0.2382
3	0.0261
4	2.9222×10^{-4}



As we expected, Newton's method has a quadratic convergence near the actual root.

$\boxed{5}$ (a) A quadrature of 3^{rd} order integrates exactly polynomials of degree ≤ 3. Newton-Cotes quadratures would need 4 points.

Gauss quad. would need $\rightarrow 2n+1 = 3 \rightarrow n = 1 \rightarrow n+1 = 2$ points

so the minimum number of points is 2.

We want to compute $I = \int_0^1 f(x)\,dx$ and we can use Gauss-Legendre quadrature

$$\int_{-1}^{1} f(z)\,dz \simeq \sum_{i=0}^{n=1} w_i f(z_i)$$ which already

has the weights tabulated

$n=1$ (order 3) $\Rightarrow$ $z_0 = -\sqrt{3}/3$, $z_1 = \sqrt{3}/3$

$w_0 = 1$, $w_1 = 1$

so the idea is to change from $[-1,1]$ to $[0,1]$

$$x_i = \frac{b-a}{2} z_i + \frac{a+b}{2} = \frac{1}{2} z_i + \frac{1}{2} \rightarrow dx = \frac{1}{2} dz$$

$$\Rightarrow \int_0^1 f(x)dx = \frac{1}{2}\int_{-1}^{1} f(z)\,dz \approx \frac{1}{2}\sum_{i=0}^{n=1} w_i f(z_i)$$

Points & weights:

$$x_0 = \frac{1}{2} z_0 + \frac{1}{2} = \frac{1}{2}\left(-\frac{\sqrt{3}}{3}\right) + \frac{1}{2} = \frac{3-\sqrt{3}}{6} \approx 0.211 \quad , w_0 = 1$$

$$x_1 = \frac{1}{2} z_1 + \frac{1}{2} = \frac{1}{2}\left(\frac{\sqrt{3}}{3}\right) + \frac{1}{2} = \frac{3+\sqrt{3}}{6} \approx 0.7886 \quad , w_1 = 1$$

(b) $x_0 = 1/4$, $x_1 = 1/2$, $x_2 = 3/4$, $x_3 = 1 \Rightarrow \{x_i\}_0^n$

Points are predetermined $\Rightarrow$ Newton–Cotes quadratures

In order to integrate a 3rd polynomial (3rd order quad.) exactly I can use Simpson's rule $(n=2) \Rightarrow$ ~~when~~ when n is even, I know that I get an extra degree of the polynomial which is integrated exactly.

$n=2 \Rightarrow 3$ points $\rightarrow x_0 = 1/4$, $x_1 = 1/2$, $x_2 = 3/4$

and use open formula of Simpson's rule.

$$I = \frac{4h}{3}\left[2f(x_0) - f(x_1) + 2f(x_2)\right] + E \quad : h = 1/4 \quad : E = 0$$

$$x_0 = 1/4 \rightarrow w_0 = \frac{8h}{3} = \frac{2}{3}$$

$$x_1 = 1/2 \rightarrow w_1 = -\frac{4h}{3} = \frac{-1}{3}$$

$$x_2 = 3/4 \rightarrow w_2 = \frac{8h}{3} = \frac{2}{3}$$

[6] (a) Using Gaussian quadratures with $n+1$ points, polynomials of degree up to $2n+1$ can be integrated exactly

(b) $n=2 \rightarrow 2n+1=5 \Rightarrow$ polynomials up to 5th degree can be integrated exactly. Thus

i) $\int_0^1 \sin x \, dx \Rightarrow$ NO (it's not a polynomial)

ii) $\int_0^1 x^3 \, dx \Rightarrow$ YES (degree <5)

iii) $\int_0^1 x^4 \, dx \Rightarrow$ YES (degree <5)

iv) $\int_0^1 x^{5.5} \, dx \Rightarrow$ NO (degree >5)

[7] Compute $\int_0^1 12x \, dx$, $\int_0^1 (5x^3+2x)\,dx$

i) Trapezoidal rule with 2 intervals (Trapezoidal $\Rightarrow n=1$)

$$\int_0^1 12x \, dx = \int_0^{0.5} 12x \, dx + \int_{0.5}^1 12x \, dx = I_1 + I_2 =$$

$$= \frac{h}{2}[f(0)+f(0.5)] + \frac{h}{2}[f(0.5)+f(1)] = \frac{0.5}{2}(0+6) + \frac{0.5}{2}(6+12) = 6$$

which coincides with exact solution

$$\int_0^1 (5x^3+2x)\,dx = \int_0^{0.5}(5x^3+2x)\,dx + \int_{0.5}^1 (5x^3+2x)\,dx = I_1 + I_2 =$$

$= \frac{h}{2}[f(0)+f(0.5)] + \frac{h}{2}[f(0.5)+f(1)] = \frac{0.5}{2}[0+\frac{13}{8}] + \frac{0.5}{2}[\frac{13}{8}+7]$

$= \frac{41}{16}$

Analytical: $\int_0^1 (5x^3+2x)\,dx = \frac{5x^4}{4} + x^2 \Big|_0^1 = \frac{9}{4}$

Analytical = Approx + E $\rightarrow$ E = -0.3125 (error with trapezoidal rule)

ii) Simpson's rule over 2 uniform intervals

$n=2 \rightarrow$ integrates up to 3rd degree $\rightarrow$ we expect exact values

$I_1 = \int_0^1 12x\,dx = \frac{h}{3}[f(x_0=0) + 4f(x_1=0.5) + f(x_2=1)] =$

$= \frac{0.5}{3}[0+4(6)+12] = 6$

$I_2 = \int_0^1 (5x^3+2x)\,dx = \frac{h}{3}[f(x_0=0) + 4f(x_1=0.5) + f(x_2=1)] =$

$= \frac{0.5}{3}[0+4(\frac{13}{8})+7] = \frac{9}{4}$

and both coincide with exact values as expected.

$\boxed{10}$ Solve $\int_0^1 \int_0^1 (9x^3 + 8x^2)(y^3 + y)\, dx\, dy$ using Simpson's rule in each direction

<u>Solution:</u>

- Simpson's rule $\Rightarrow n=2 \Rightarrow$ it integrates exactly up to 3rd degree polynomials so we expect to get the exact value of the integral

$$I = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

First, we integrate with respect to x : $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$

$$I_1 = \frac{0.5}{3}\left(f(0) + 4f(0.5) + f(1)\right) = \frac{0.5}{3}\left(0 + 4\left[\frac{25}{8}(y^3+y)\right]\right.$$

$$\left. + 17(y^3+y)\right] = \frac{25}{12}(y^3+y) + \frac{17}{6}(y^3+y) = \frac{59}{12}(y^3+y)$$

Now, integrate with respect to y : $y_0 = 0$, $y_1 = 0.5$, $y_2 = 1$

$$I = \frac{59}{12}\left\{\frac{h}{3}\left[f(0) + 4f(0.5) + f(1)\right]\right\} = \frac{59}{12}\left\{\frac{0.5}{3}\left[0 + 4\cdot\frac{5}{8} + 2\right]\right\} = \frac{59}{16}$$

$$I = \frac{59}{16}$$

- Analytically

$$\int_0^1\int_0^1 (9x^3+8x^2)(y^3+y)\,dx\,dy = \int_0^1 \left[\frac{9x^4}{4} + \frac{8x^3}{3}\right]_0^1 (y^3+y)\,dy$$

$$= \int_0^1 \left[\frac{9}{4} + \frac{8}{3}\right](y^3+y)\,dy = \frac{59}{12}\int_0^1 (y^3+y)\,dy = \frac{59}{12}\left[\frac{y^4}{4} + \frac{y^2}{2}\right]_0^1 =$$

$= \frac{59}{12}\cdot\frac{3}{4} = \frac{59}{16}$ which coincides with the numerical solution as expected