Finite differenced excercise

2| consider the differential equation

$$u_t + a u_x = 0, \quad x \in (0,1), \quad t \geq 0, \quad a > 0$$

with initial condition

$$u(x,0) = \sin(2\pi x)$$

And periodic bc - that is

$$u(0,t) = u(1,t)$$

(a) propose an implicit finite difference scheme, with first order in time & space, for the discretization of 3, Justify the selection of the approximation for the spatial derivative

Ans

$$\left.\frac{du}{dt}\right| = \frac{u_i^{n+1} - u_i^n}{\Delta t} \qquad \left.\frac{du}{dx}\right| = \frac{u_{i+1}^{n+1} - u_{i}^{n+1}}{\Delta x}$$

$$\frac{du}{dt} + \alpha \frac{du}{dx} = 0 \Rightarrow \frac{u_i^{n+1} - u_i^n}{\Delta t} + \alpha \frac{u^{n+1} - u^{n+1}}{\Delta x} = 0$$

$$\hookrightarrow u_i^{n+1} - u_i^n = -\frac{\Delta t \alpha}{\Delta x}\left[u_{i+1}^{n+1} - u_i^{n+1}\right]$$

let

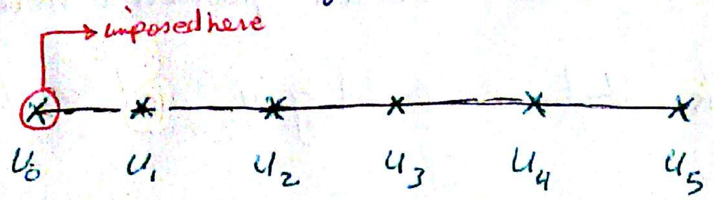
$$c = \frac{\Delta t \alpha}{\Delta x}$$

$$\boxed{[1-c]\, u_i^{n+1} + c\, u_{i-1}^{n+1} = u_i^n}$$

If this method was to be solved explicitly, choosing a proper spacial approxi (upwind in this question) would have played a crucial role in determining the stability of the function, because v is -ve the information is travelling from right to left. That's why upwind chosen

b. write down a system of eq. to solve for each time step u_i^{n+1}

Ans. we will assume a system discretization of 5 nodes



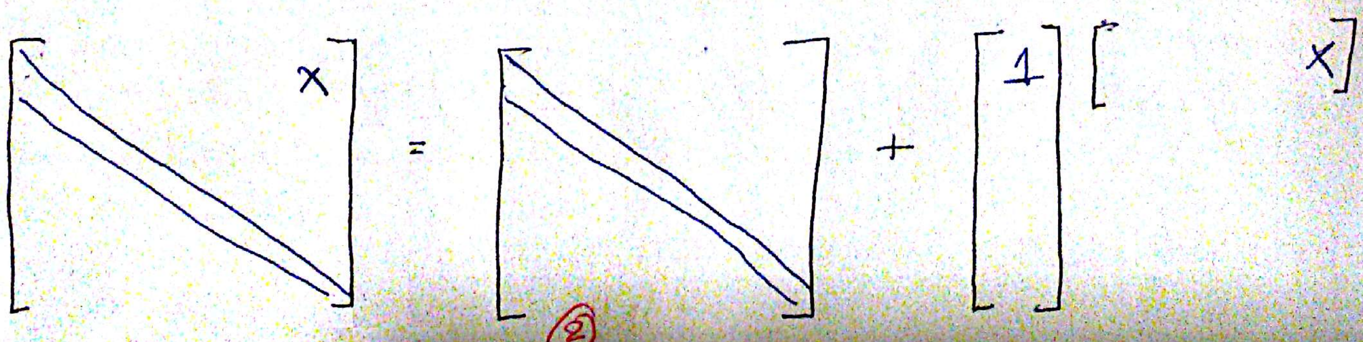
The periodic boundary conditions is stated as $U(0,t) = U(1,t)$ and the 1st node U_0 equals to the last node U_5, and the equation is imposed at U_1 but in order to determine U_1 we need U_{i-1} in this case it is U so the value at node U_1 at U_0. and due to the periodic BC we have $U_1 = U_5$ hence $c\, U^n = U$ and that can be seen in the matrix in red.

$$\begin{bmatrix} 1-c & 0 & 0 & 0 & c & 0 \\ c & 1-c & 0 & 0 & 0 & 0 \\ 0 & c & 1-c & 0 & 0 & 0 \\ 0 & 0 & c & [1-c] & 0 & 0 \\ 0 & 0 & 0 & c & [1-c] & 0 \\ 0 & 0 & 0 & & c & [1-c] \end{bmatrix} \begin{bmatrix} u_0^{n+1} \\ u_1^{n+1} \\ u_2^{n+1} \\ u_3^{n+1} \\ u_4^{n+1} \\ u_5^{n+1} \end{bmatrix} = \begin{bmatrix} U_0^n \\ U_1^n \\ U_2^n \\ U_3^n \\ U_4^n \\ U_5^n \end{bmatrix}$$

c. Direct method $\rightarrow$ Sherman - Morrison method

Iterative method $\rightarrow$ Gauss- Siedle $\qquad A = A^* + U V^T$

d.



4.

$$u_t = \nu u_{xx} + \sigma u$$

$$BC \rightarrow \begin{cases} u(0,t) = 0 \\ u_x(1,t) = 0 \end{cases}$$

$$IC \rightarrow \begin{cases} 0 & x < 1/4 \\ 4x-1 & 1/4 \leq x < 1/2 \\ -4x+3 & 1/2 \leq x < 3/4 \\ 0 & 3/4 \leq x \end{cases}$$

Answer

a. The FTCS finite difference scheme is proposed for this problem.

%

$$u_t = \frac{du}{dt}\Big|_i^n = \frac{u_i^{n+1} - u_i^n}{\Delta t}$$

$$u_{xx} = \frac{d^2u}{dx^2}\Big|_i^n = \frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{(\Delta x)^2}$$

Ghost node

$$u = \frac{du}{dx}\Big|_{i=4}^n = \frac{u_5^n - u_3^n}{\Delta x} = 0 \Rightarrow u_5^n = u_3^n \Rightarrow u_{i+1} = u_{i-1} \text{ when } i = 4$$

% The numerical treatment of the pde:

$$* \; u_t = \nu u_{xx} + \sigma u \Rightarrow \frac{u_i^{n+1} - u_i^n}{\Delta t} = \nu \frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{(\Delta x)^2} + \sigma u_i^n$$

$$* \; u_i^{n+1} = \frac{\Delta t \, \nu}{(\Delta x)^2}\left[u_{i-1}^n - 2u_i^n + u_{i+1}^n\right] + \sigma \Delta t (u_i^n) + u_i^n$$

% let $\boxed{r = \frac{\Delta t \, \nu}{(\Delta x)^2}}$ + $\boxed{q = \sigma \Delta t}$

$$= r\left[u_{i-1}^n - 2u_i^n + u_{i+1}^n\right] + q\, u_i^n + u_i^n$$

b-

For $\sigma = 0$

The equation becomes a 1-D parabolic equation where BTCS or Crank-Nichelson could give good results.

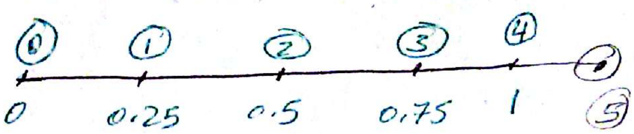
For $v = 0$

The equation becomes a simple ODE that could be simply solved using a simple finite difference scheme or using Runge-Kutta c- $u_i^{n+1} = r\, u_{i-1}^n - \underline{2r\, u_i^n} + r\, u_{i+1}^n + \underline{q\, u_i^n} + \underline{u_i^n}$

$$= r \cdot u_{i-1}^n + [1 + q - 2r]\, u_i^n + r\, u_{i+1}^n$$

$$\boxed{u_i^{n+1} = r\, u_{i-1}^n + [1+q-2r]\, u_i^n + r\, u_{i+1}^n}$$

% spacial discretization



$i = 1, 2, 3$

% Initial conditions

$u_1^0 \to$ Node 1 $(1/4 \leq x < 1/2) = 4x - 1 = 4(0.25) - 1 = \boxed{0}$

$u_2^0 \to$ Node 2 $(1/2 \leq x < 3/4) = -4x + 3 = -4(0.5) + 3 = \boxed{1}$

$u_3^0 \to$ Node 3 $(3/4 \leq x) = \boxed{0}$ $\qquad u_5 = u_3$

$u_0^0 \to$ Node 0 $(x < 1/4) = \boxed{0}$

$u_4^0 \to$ Node 4 $(3/4 \leq x) = \boxed{0}$

% Boundary conditions

$u_0^n \to \boxed{0}$

$\left.\frac{du}{dx}\right|_4^n \to \boxed{0}$

% constant Calculation:

$\boxed{\nu = 0.1}$ $\boxed{\sigma = -0.1}$ $\boxed{\Delta x = 0.25}$ $\boxed{\Delta t = 0.1}$

$$r = \frac{\Delta t \, \nu}{(\Delta x)^2} = \frac{(0.1)(0.1)}{(0.25)^2} = \boxed{0.16}$$

$$q = \sigma \Delta t = (-0.1)(0.1) = \boxed{-0.01}$$

First time step: [t = 0.1]

$r = 0.16$, $q = -0.01$, $u_i^{n+1} = r u_{i-1}^n + [\overbrace{1+q-2r}^{0.67}] u_i^n + r u_{i+1}^n$

$u_0^{0.1} = \boxed{0}$

$u_1^{0.1} = 0.16 u_0^0 + 0.67\, u_1^0 + 0.16\, u_2^0$

$= 0.16(0) + 0.67(0) + 0.16(1) = \boxed{0.16}$

$u_2^{0.1} = 0.16\, u_1^0 + 0.67\, u_2^0 + 0.16\, u_3^0$

$= 0.16(0) + 0.67(1) + 0.16(0) = \boxed{0.67}$

$u_3^{0.1} = 0.16\, u_2^0 + 0.67\, u_3^0 + 0.16\, u_4^0$

$= 0.16(1) + 0.67(0) + 0.16(0) = \boxed{0.16}$

$u_4^{0.1} = 0.16\, u_3^0 + (0.67)\, u_4^0 + 0.16\, \cancel{u_5} \rightarrow u_3$ (Neuman BC)

$= 0.16(0) + 0.67(0) + 0.16(0) = \boxed{0}$

Second time step [t = 0.2]

$r = 0.16$, $q = -0.01$, $u_i^{n+1} = r u_{i-1}^{n} + [0.67] \cdot u_i^n + r u_{i+1}^{n}$

$u_0^{0.1} = 0$

$u_1^{0.1} = 0.16$

$u_2^{0.1} = 0.67$

$u_3^{0.1} = 0.16$

$u_4^{0.1} = 0$

$$u_0^{0.2} = \boxed{0}$$

$$u_1^{0.2} = 0.16\, u_0^{0.1} + 0.67\, u_1^{0.1} + 0.16\, u_2^{0.1}$$

$$0.16(0) + 0.67(0.16) + 0.16(0.67) = \boxed{0.2144}$$

$$=$$

$$u_2^{0.2} = 0.16\, u_1^{0.1} + 0.67\, u_2^{0.1} + 0.16\, u_3^{6.1}$$

$$= 0.16(0.16) + 0.67(0.67) + 0.16(0.16) = \boxed{0.5001}$$

$$u_3^{0.2} = 0.16\, u_2^{0.1} + 0.67\, u_3^{0.1} + 0.1\, u_4^{0}$$

$$= 0.16(0.67) + 0.67(0.16) + 0.1(0) = \boxed{0.2144}$$

$$u_4^{0} = 0.16\, u_3^{0.} + 0.67\, u_4^{0.1} + 0.16\, \cancel{u_5} \rightarrow u_3^{0.1} \text{ (Neuman BC)}$$

$$= 0.16(0.16) + 0.67(0) + 0.16(0.16) = \boxed{0.0512}$$

The results obtained and can be seen graphed the next page shows a satisfying solution for this diffusion reaction problem due to the parabolic shape of the curve which coincides with the real analyt soli of this eq. further more the evolution of ~~the~~ curve with time gives us a good indication.

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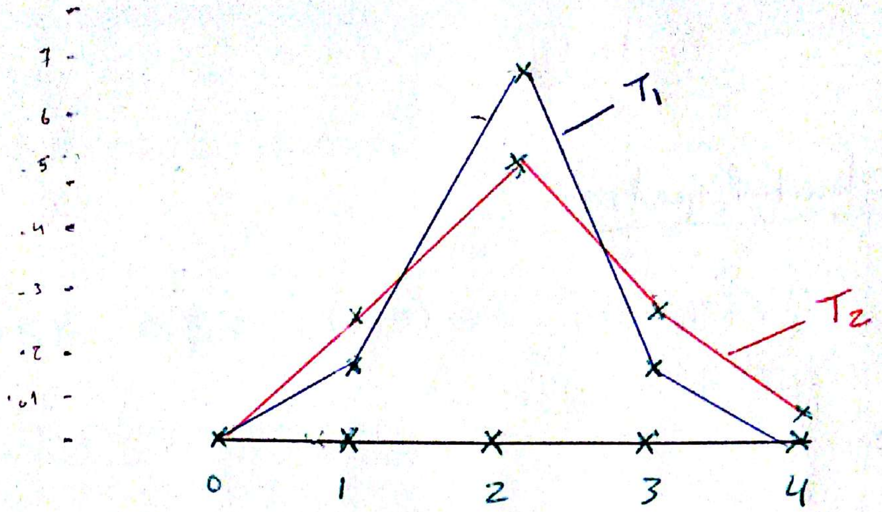

T_1
T_2
0
1
2
3
4

d- The BTCS finite difference scheeme is proposed

% FD approximation

$$u_t = \frac{du}{dt}\Big|_i^n = \frac{u_i^{n+1} - u_i^n}{\Delta t}$$

$$u_{xx} = \frac{d^2u}{dt^2}\Big|_i^{n+1} = \frac{u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}}{(\Delta x)^2}$$

% Treatment of the Boundary Conditions & Initial conditions

Boundary conditions are defined at 2 nodes, U_0 & U_4

$U_0 = 0$ at each time step

$$\frac{d^2u}{dx}\Big|_4^{n+1} = \frac{u_5 - u_3}{\Delta x} = 0 \Rightarrow \quad u_5 = u_3$$ Only at eq. imposed

@ node 4

% BTCS scheme is obt. by sub. the approximations in the wave eq.

$$u_t = Vu_{xx} + \sigma u \Rightarrow \frac{u_i^{n+1} - u_i^n}{\Delta t} = [V] \frac{u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}}{(\Delta x)^2} + [\sigma]\, u_i^n$$

$$-r\, u_{i-1}^{n+1} + [1+2r]\, u_i^{n+1} - r\, u_{i+1}^{n+1} = [1+q]\, u_i^n$$

where i is from $1 \rightarrow m$ not m.

Only at the eq. is aimposeded at Nonde no. 4 due to Neuman Boundary conditions

$$-2r\, u_{i-1}^{n+1} + [1+2r]\, u_i^{n+1} = [1+q]\, u_i^n$$

where $U_5^n = U_3^n \rightarrow U_{i+1}^n = U_{i-1}^n$ where $i = 4$

$$
\begin{bmatrix} [1+2r] & -r & 0 & 0 \\ -r & [1+2r] & -r & 0 \\ 0 & -r & [1+2r] & -r \\ 0 & 0 & \boxed{-2r} & [1+2r] \end{bmatrix}
\begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ u_3^{n+1} \\ u_4^{n+1} \end{bmatrix}
=
\begin{bmatrix} u_1^{n} \\ u_2^{n} \\ u_3^{n} \\ u_4^{n} \end{bmatrix}
* [1xq] +
\begin{bmatrix} -u_0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Neumann BC.