Numerical Methods for PDE's, ODE's

1. The motion of a non-frictional pendulum is governed by the Ordinary Differential Equation (ODE)

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

where θ is the angular displacement, $L = 1$ m is the pendulum length and the gravity acceleration is $g = 9.8$ m/s^2.

The position and velocity at time $t = 1$ s are known.

$$\theta(1) = 0.4 \text{ rad} \quad ; \quad \frac{d\theta}{dt}(1) = 0 \text{ rad/s}$$

a) Solve the initial boundary value problem in the interval $(0, 1)$ using a second-order Runge-Kutta method to determine the initial position at $t = 0$ s, with 2 and 4 time steps.

- First of all, we have to transform the second-order derivative in two first-order derivative.

$$\Rightarrow \bar{y} \stackrel{\text{def}}{=} \begin{Bmatrix} \theta \\ \frac{d\theta}{dt} \end{Bmatrix} = \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} ; \Rightarrow \frac{d\bar{y}}{dt} = \begin{Bmatrix} d\theta/dt \\ d^2\theta/dt^2 \end{Bmatrix} = \begin{Bmatrix} y_2 \\ -\frac{g}{L} y_1 \end{Bmatrix} \stackrel{\text{def}}{=} \bar{f}$$

$$\Rightarrow \bar{\alpha} = \begin{Bmatrix} \theta(1) \\ \frac{d\theta}{dt}(1) \end{Bmatrix} = \begin{Bmatrix} 0.4 \\ 0 \end{Bmatrix} = Y_0$$

- The second-order Runge-Kutta method (Heunn method) is defined as:

$$Y_{i+1} = Y_i + \frac{h}{2}(K_1 + K_2) \text{ with } \begin{cases} K_1 = f(x_i, Y_i) \\ K_2 = f(x_i + h, Y_i + hK_1) \end{cases}$$

- Let's compute the method for both time steps:

• **m = 2**

In this case $h = \frac{b-a}{m} = \frac{0-1}{2} = -\frac{1}{2}$

• Y_1:

$$\Rightarrow K_1 = f(x_0, Y_0) = f\left(1, \begin{Bmatrix} 0.4 \\ 0 \end{Bmatrix}\right) = \begin{Bmatrix} 0 \\ -\frac{9.8}{1} \cdot 0.4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -3.92 \end{Bmatrix};$$

$$\Rightarrow Y_0 + hK_1 = \begin{Bmatrix} 0.4 \\ 0 \end{Bmatrix} - \frac{1}{2}\begin{Bmatrix} 0 \\ -3.92 \end{Bmatrix} = \begin{Bmatrix} 0.4 \\ 1.96 \end{Bmatrix}$$

$$\Rightarrow K_2 = f(x_0+h, Y_0+hK_1) = f\left(\frac{1}{2}, \begin{Bmatrix} 0.4 \\ 1.96 \end{Bmatrix}\right) = \begin{Bmatrix} 1.96 \\ -\frac{9.8}{1} \cdot 0.4 \end{Bmatrix} = \begin{Bmatrix} 1.96 \\ -3.92 \end{Bmatrix}$$

$$\Rightarrow Y_1 = Y_0 + \frac{h}{2}[K_1 + K_2] = \begin{Bmatrix} 0.4 \\ 0 \end{Bmatrix} - \frac{1}{4}\left[\begin{Bmatrix} 0 \\ -3.92 \end{Bmatrix} + \begin{Bmatrix} 1.96 \\ -3.92 \end{Bmatrix}\right] = \begin{Bmatrix} -0.09 \\ 1.96 \end{Bmatrix}$$

• Y_2:

$$\Rightarrow K_1 = f(x_1, Y_1) = f\left(\frac{1}{2}, \begin{Bmatrix} -0.09 \\ 1.96 \end{Bmatrix}\right) = \begin{Bmatrix} 1.96 \\ -\frac{9.8}{1} \cdot (-0.09) \end{Bmatrix} = \begin{Bmatrix} 1.96 \\ 0.882 \end{Bmatrix};$$

$$\Rightarrow Y_1 + hK_1 = \begin{Bmatrix} 0.09 \\ 1.96 \end{Bmatrix} - \frac{1}{2}\begin{Bmatrix} 1.96 \\ 0.882 \end{Bmatrix} = \begin{Bmatrix} -1.07 \\ 1.519 \end{Bmatrix}$$

$$\Rightarrow K_2 = f(x_1+h, Y_1+hK_1) = f\left(0, \begin{Bmatrix} -1.07 \\ 1.519 \end{Bmatrix}\right) = \begin{Bmatrix} 1.519 \\ -\frac{9.8}{1} \cdot (-1.07) \end{Bmatrix} = \begin{Bmatrix} 1.519 \\ 10.486 \end{Bmatrix}$$

$$\Rightarrow Y_2 = Y_1 + \frac{h}{2}[K_1 + K_2] = \begin{Bmatrix} -0.09 \\ 1.96 \end{Bmatrix} - \frac{1}{4}\left[\begin{Bmatrix} 1.96 \\ 0.882 \end{Bmatrix} + \begin{Bmatrix} 1.519 \\ 10.486 \end{Bmatrix}\right] = \begin{Bmatrix} 0.078 \\ -0.882 \end{Bmatrix}$$

• So,

$$\Rightarrow \boxed{\bar{y}(0) \approx Y_2 = \begin{Bmatrix} -0.9598 \\ -0.882 \end{Bmatrix}} \longrightarrow \boxed{\theta(0) = -0.9598 \text{ rad}}$$

• **m = 4**

In this case $h = \frac{b-a}{m} = \frac{0-1}{4} = -\frac{1}{4}$

• Y_1:

$$\Rightarrow K_1 = f(x_0, Y_0) = f\left(1, \begin{Bmatrix} 0.4 \\ 0 \end{Bmatrix}\right) = \begin{Bmatrix} 0 \\ -\frac{9.8}{1} \cdot 0.4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -3.92 \end{Bmatrix}$$

$$\Rightarrow Y_0 + hK_1 = \begin{Bmatrix} 0.4 \\ 0 \end{Bmatrix} - \frac{1}{4}\begin{Bmatrix} 0 \\ -3.92 \end{Bmatrix} = \begin{Bmatrix} 0.4 \\ 0.98 \end{Bmatrix}$$

$$\Rightarrow K_2 = f(x_0+h, Y_0+hK_1) = f\left(\frac{3}{4}, \begin{Bmatrix} 0.4 \\ 0.98 \end{Bmatrix}\right) = \begin{Bmatrix} 0.98 \\ -3.92 \end{Bmatrix}$$

$$\Rightarrow Y_1 = Y_0 + \frac{h}{2}[K_1+K_2] = \begin{Bmatrix} 0.4 \\ 0 \end{Bmatrix} - \frac{1}{8}\left[\begin{bmatrix} 0 \\ -3.92 \end{bmatrix} + \begin{Bmatrix} 0.98 \\ -3.92 \end{Bmatrix}\right] = \begin{Bmatrix} 0.2775 \\ 0.98 \end{Bmatrix}$$

- Y_2

$$\Rightarrow K_1 = f(x_1, Y_1) = f\left(\frac{3}{4}, \begin{Bmatrix} 0.2775 \\ 0.98 \end{Bmatrix}\right) = \begin{Bmatrix} 0.98 \\ -2,7195 \end{Bmatrix}$$

$$\Rightarrow K_2 = f(x_1+h, Y_1+hK_1) = \begin{Bmatrix} 1.6599 \\ -0.3185 \end{Bmatrix}$$

$$\Rightarrow Y_2 = Y_1 + \frac{h}{2}[K_1+K_2] = \begin{Bmatrix} 0.2775 \\ 0.98 \end{Bmatrix} - \frac{1}{8}\left[\begin{Bmatrix} 0.98 \\ -2.7195 \end{Bmatrix} + \begin{Bmatrix} 1.6599 \\ -0.3185 \end{Bmatrix}\right] = \begin{Bmatrix} -0.0525 \\ 1.3598 \end{Bmatrix}$$

- By doing the same procedure, it is obtained:

$$\Rightarrow Y_3 = \begin{Bmatrix} -0.3763 \\ 0.8147 \end{Bmatrix} \quad ; \quad Y_4 = \begin{Bmatrix} -0.4648 \\ -0.3568 \end{Bmatrix}$$

- So,

$$\Rightarrow y(0) \approx Y_4 = \begin{Bmatrix} -0.4648 \\ -0.3568 \end{Bmatrix} \Rightarrow \boxed{\theta(0) \approx -0.4648 \text{ rad}}$$

b) Using the approximations obtained in a), compute an approximation of the relative error in the solution with 2 steps

- To obtain an approximation of the relative error we can compute with MatLab the solution for a high number of steps and consider it as the real solution:

$$\rightarrow \bar{y}(0) \approx \begin{Bmatrix} -0.4 \\ 0.0139 \end{Bmatrix} \rightarrow \underline{\theta(0) = -0.4}$$

- So the relative error in calculating the initial position with 2 steps

$$\Rightarrow \boxed{e_{r_2} =} \left| \frac{-0.9598+0.4}{-0.4} \right| \boxed{= 1,4}$$

c) Propose a time step h to obtain an approximation with a relative error three orders of magnitude smaller

- For Heunn's method, which is a second-order method

$$\Rightarrow E_h = C \cdot h^{-2}$$

- We want to obtain a time step h^* such that

$$\Rightarrow E_h^* = 10^{-3} \cdot E_h \quad ; \quad E_h^* = C \cdot h^{*-2}$$

- So, at the end:

$$\Rightarrow \frac{E_h^*}{E_h} = 10^{-3} = \frac{C \cdot h^{*-2}}{C \cdot h^{-2}} = \frac{h^{+2}}{h^{*+2}} \Longleftrightarrow h^{*-2} = 10^{-3} \cdot h^{-2}$$

$$\Longleftrightarrow h^{*-2} = 10^{-3} \cdot \frac{1}{(-0.5)^2} = \frac{1}{250} \Longleftrightarrow \frac{1}{h^{*2}} = \frac{1}{250} \Rightarrow h^* \Longrightarrow$$

- We arrive to the formula

$$\Rightarrow \boxed{h^* =} \left(\frac{tol}{E_h}\right)^{1/p+1} \cdot h = \left(10^{-3}\right)^{1/2} (-0,5) \boxed{= -0,016}$$

2. Consider the initial value problem

$$\frac{dy}{dx} = y - x^2 + 1 \qquad x \in (0, 1)$$

$$y(0) = 1$$

a) Solve the initial value problem using the Euler method with step $h = 0.25$.

- Euler method can be written as

$$Y_{i+1} = Y_i + h \cdot f(x_i, Y_i) \quad , \text{with } Y_0 = \{1\}$$

$$\Rightarrow Y_1 = 1 + 0.25 \cdot (1 - (0)^2 + 1) = 1,5000$$

$$\Rightarrow Y_2 = 1.5 + 0.25(1.5 - (0.25)^2 + 1) = 2,1094$$

$$\Rightarrow Y_3 = 2.1094 + 0.25(2.1094 - (0.5)^2 + 1) = 2.8242$$

$$\Rightarrow Y_4 = 2,8242 + 0.25(2.8242 - (0.75)^2 + 1) = 3,6396$$

• So,

$$\Longrightarrow \boxed{y(1) \approx Y_4 = 3,6396}$$

b) Compute the solution using the Heunn method with a step h such that the computational cost is equivalent to the computational cost in a)

• The computational cost can be seen as the number of function evaluations. For each iteration, Heunn needs 2 evaluations and Euler only one. So we will use $h = 0.5$.

• The Heunn method is defined as:

$$Y_{i+1} = Y_i + \frac{h}{2}\left[K_1 + K_2\right] \text{ with } \begin{cases} K_1 = f(x_i, Y_i) \\ K_2 = f(x_i + h, Y_i + hK_1) \end{cases}$$

• Let's compute the method:

• Y_1

$\Rightarrow K_1 = f(x_0, Y_0) = f(0, 1) = 2$

$\Rightarrow Y_0 + hK_1 = 1 + 0.5 \cdot 2 = \cancel{1.5}\ 2$, $K_2 = f(0.5, 2) = 2.75$

$\Rightarrow Y_1 = Y_0 + 0.25 \cdot (2 + 2.75) = 2.1875$

• Y_2

$\Rightarrow K_1 = f(x_1, Y_1) = f(0.5, 2.1875) = 2.9375$

$\Rightarrow Y_1 + hK_1 = 2.1875 + 0.5 \cdot 2.9375 = 3.6563$

$\Rightarrow K_2 = f(1, 3.6563) = 3.6563$

$\Rightarrow Y_2 = Y_1 + 0.25 \cdot (2.9375 + 3.6563) = 3.8359$

• So,

$$\Longrightarrow \boxed{y(1) \approx Y_2 = 3.8359}$$

Note that the analytical solution of the initial value problem is a second degree polynomial.

c) Compute the pure interpolation polynomial that fits the results in b)

- Through Heunn's method, it has been obtained that:

$\Rightarrow y(0) = 1$; $y(0.5) \approx 2.1875$; $y(1) = 3.8359$

- We want to approximate $y(x) \approx p(x)$, and knowing that the solution is a second degree polynomial, the Uniqueness theorem guarantees us that only one polynomial pass exactly through these 3 points.

$$\Rightarrow p(x) = ax^2 + bx + c \quad \text{such that} \begin{cases} p(0) = c = 1 \\ p(0.5) = \frac{1}{4}a + \frac{1}{2}b + 1 = 2.1875 \\ p(1) = a + b + 1 = 3.8359 \end{cases}$$

$$\Leftrightarrow \begin{cases} \frac{1}{4}a + \frac{1}{2}b = \frac{19}{16} \\ a + b = 2.8359 \rightarrow b = 2.8359 - a \end{cases} \longrightarrow \frac{1}{4}a + 1.418 - \frac{1}{2}a = \frac{19}{16}$$

$$\Rightarrow -\frac{1}{4}a = -0.2305 \rightarrow a = 0.922 \rightarrow b = 1.9139$$

- So the pure interpolation polynomial that fits the results in b)

$$\Rightarrow \boxed{y(x) \approx p(x) = 0.922x^2 + 1.9139x + 1}$$

d) Which approximation criterion do you recommend to fit the results obtained in a)? Compare the polynomial approximation with the proposed criterion and compare the results with the polynomial obtained in c)

- In this case, there are 5 data:

$\rightarrow y(0) = 1$; $y(0.25) = 1.5$; $y(0.5) = 2.1094$

$y(0.75) = 2.8242$; $y(1) = 3.6396$

If we want to fit all the data, we have to use parabolic splines between 0 and 0.5 and 0.5 and 1. And we have to impose that there must be continuity and derivability at 0.5.

- $S_1(x) = a_1 x^2 + b_1 x + c_1$ such that
$$\begin{cases} S_1(0) = c_1 = 1 \\ S_1(0.25) = \frac{a_1}{16} + \frac{b_1}{4} + 1 = 1.5 \\ S_1(0.5) = \frac{a_1}{4} + \frac{b_1}{2} + 1 = 2.1094 \end{cases}$$

Solving it,

$$\rightarrow S_1(x) = 0.8752\, x^2 + 1.7812\, x + 1 \qquad \text{for } 0 \leq x \leq 0.5$$

- $S_2(x) = a_2 x^2 + b_2 x + c_2$ such that
$$\begin{cases} S_2(0.5) = \frac{a_2}{4} + \frac{b_2}{2} + c_2 = 2.1094 \\ S_2(0.75) = \frac{9a_2}{16} + \frac{3b_2}{4} + c_2 = 2.8242 \\ S_2(1) = a_2 + b_2 + c_2 = 3.6396 \end{cases}$$

Solving it,

$$\rightarrow S_2(x) = 0.8048\, x^2 + 1.8532\, x + 0.9816 \qquad \text{for } 0.5 \leq x \leq 1$$

So,

$$\boxed{y(x) \approx S(x) = \begin{cases} 0.8752\, x^2 + 1.7812\, x + 1 & \text{for } 0 \leq x < 0.5 \\ 0.8048\, x^2 + 1.8532\, x + 0.9816 & \text{for } 0.5 \leq x \leq 1 \end{cases}}$$

As we can see splines allow us to impose more points by splitting the domain while polynomial fitting can not use more than n+1 points to obtain ~~a polynomial~~ of second-order degree polynomial.

3. The ordinary differential equation

$$\frac{dy}{dx} = f(x,y)$$

is defined over the domain (0,1), and is to be solved numerically subject to the initial condition $y(0) = 1$, where $y(x)$ is the exact solution. The forward Euler method for integrating the above differential equation is written as

$$Y_{i+1} = Y_i + h \cdot f(x_i, Y_i)$$

where Y_i denotes the discrete solution at node i, with position x_i, of a uniform grid of nodes of constant grid interval size h and $x_{i+1} = x_i + h$.

a) Using a Taylor series expansion, deduce the leading truncation error of the scheme. Is the method consistent? Explain your answer.

- Applying Taylor series to y_{i+1}

$$\Rightarrow y_{i+1} = y_i + h \cdot \frac{dy}{dx}(x_i) + O(h^2)$$

$$\Leftrightarrow \frac{dy}{dx}(x_i) = \frac{y_{i+1} - y_i}{h} - \tau_i(h), \quad \text{with } \boxed{\tau_i(h) = O(h)}$$

is the truncation error

- Replacing $\frac{dy}{dx}$ by $f(x, y)$ ~~and~~ the equation yields,

$$\rightarrow y_{i+1} = y_i + h \cdot f(x_i, y_i) + h \cdot \tau_i(h)$$

- Neglecting the truncation error, we end up by obtaining:

$$\Rightarrow Y_{i+1} = Y_i + h \cdot f(x_i, Y_i)$$

- A method is said to be consistent if it verifies

$$\max_{0 \le i \le m} \tau_i(h) \rightarrow 0 \quad \text{when } h \rightarrow 0$$

- In this method, $\tau_i(h) = O(h)$, so it's proportional to h;

$$\Rightarrow \boxed{\lim_{h \to 0} \tau_i(h)} = \lim_{h \to 0} O(h) = \boxed{0}$$

So the method is consistent.

b) State the backward Euler method for integrating the above differential equation where $f(x,y)$ is a general non-linear function of x and y.

- In this case it is obtained the i-point based on the Taylor series in y_{i+1}:

$$\Rightarrow y_i = y_{i+1} - h\frac{dy}{dx}(x_{i+1}) + O(h^2)$$

$$\Rightarrow \frac{dy}{dx}(x_{i+1}) = \frac{y_{i+1}-y_i}{h} + \tau_i(h)$$

- Replacing $\frac{dy}{dx}(x_{i+1})$ by $f(x_{i+1}, y_{i+1})$, we obtain:

$$\rightarrow y_{i+1} = y_i + h f(x_{i+1}, y_{i+1}) + h\tau_i(h)$$

- Neglecting the truncation error, it is obtained the backward Euler method.

$$\Rightarrow \boxed{Y_{i+1} = Y_i + h\cdot f(x_{i+1}, Y_{i+1})}$$

c) Deduce the stability limits of the respective forward Euler method and backward Euler method for the model equation $dy/dx = -\lambda y$ where λ is a positive real constant.

- Stability analysis for the Euler method

$$\Rightarrow Y_{i+1} = Y_i + h\cdot f(x_i, Y_i) = Y_i + -h\lambda Y_i = (1-h\lambda)Y_i$$

$$\Rightarrow Y_{i+1} = G\cdot Y_i \quad , \text{with} \quad G = 1-\lambda h$$

- The scheme will be absolutely stable if $|G|<1$, that is

$$\Rightarrow |1-\lambda h| < 1 \Leftrightarrow \begin{cases} 1-\lambda h < 1 & \Leftrightarrow \lambda h > 0 \quad \text{and } \lambda h < 1 \\ -1+\lambda h < 1 & \Leftrightarrow \lambda h < 2 \quad \text{and } \lambda h > 1 \end{cases}$$

$$\Rightarrow \boxed{0 < \lambda h < 2}$$

- Stability analysis for the backward Euler

$$\Rightarrow Y_{i+1} = Y_i - h\lambda Y_{i+1} \Leftrightarrow (1+h\lambda) Y_{i+1} = Y_i$$

$$\Rightarrow Y_{i+1} = G \cdot Y_i \quad , \text{ with } \quad G = \frac{1}{1+h\lambda}$$

- The scheme will be absolutely stable if $|G| < 1$, that is,

$$\Rightarrow \left| \frac{1}{1+h\lambda} \right| < 1 \Leftrightarrow |1+\lambda h| > 1 \begin{cases} \lambda h > 0 & \text{if } \lambda h > -1 \\ \lambda h < -2 & \text{if } \lambda h < -1 \end{cases}$$

$$\Rightarrow \boxed{\lambda h < -2 \quad \text{or} \quad \lambda h > 0}$$

d) Use the backward Euler method to compute the numerical solution of the ordinary differential equation

$$\frac{dy}{dx} = -25 \cdot y^{3.5}$$

with initial condition $y(0) = 1$, by hand for two steps with grid interval size $h (= 1/10$. (Use 2 Newton iterations per step for this calculation)

- $Y_0 = 1$;
- Y_1:

$$\Rightarrow Y_1 = Y_0 + h \cdot f(x_1, Y_1) = Y_0 + h \cdot (-25 \cdot Y_1^{3.5}) \Rightarrow$$

- To obtain Y_1, as it is defined implicitly, we have to use Newton. Let's define:

$$\rightarrow g_1(Y_1) = Y_0 + \frac{1}{10} \cdot (-25 \cdot Y_1^{3.5}) - Y_1 = 0$$

$$\rightarrow g_1(Y_1) = 1 - \frac{5}{2} Y_1^{3.5} - Y_1 = 0$$

- As a initial approximation, it will be neglected the monomial Y_1 with respect the monomial $Y_1^{3.5}$.

$$\Rightarrow 1 = \frac{5}{2} y_1^{0\,3.5} \Rightarrow y_1^0 = \sqrt[3.5]{\frac{2}{5}} = 0,7697$$

- The derivative will be ~~needed~~ necessary

$$\Rightarrow g_1'(y_1) = -\frac{35}{4} y_1^{2.5} - 1$$

- So, by using Newton-Raphson:

$$\Rightarrow y_1^1 = y_1^0 - \frac{g_1(y_1^0)}{g_1'(y_1^0)} = 0.6309$$

$$\Rightarrow y_1^2 = y_1^1 - \frac{g_1(y_1^1)}{g_1'(y_1^1)} = \boxed{0.5965 = y_1}$$

- y_2,

$$\Rightarrow y_2 = y_1 + h \cdot f(x_2, y_2) = y_1 + h \cdot (-25 \cdot y_2^{3.5})$$

- And again, we have to define a function to be solved with Newton-Raphson.

$$\Rightarrow g_2(y_2) = y_1 + h \cdot (-25 y_2^{3.5}) - y_2 = 0$$

$$\Rightarrow g_2(y_2) = 0.5965 + \frac{1}{10} \cdot (-25 y_2^{3.5}) - y_2 = 0$$

- As a initial approximation, we can neglect y_2 with respect to y_2^{35}:

$$\Rightarrow 0.5965 = \frac{5}{2} y_2^{0\,3.5} \Rightarrow y_2^0 = 0.664$$

- The derivative will be necessary

$$\Rightarrow g_2'(y_2) = -\frac{35}{4} y_2^{2.5} - 1$$

- By using Newton,

$$\Rightarrow y_2^1 = y_2^0 - \frac{g_2(y_2^0)}{g_2'(y_2^0)} = 0.5038$$

$$\Rightarrow y_2^2 = y_2^1 - \frac{g_2(y_2^1)}{g_2'(y_2^1)} = \boxed{0.4517 = y_2}$$

e) Use the forward Euler method to compute the numerical solution of the above ordinary differential equation with same initial condition by hand for two step with grid interval size $h = 1/10$.

- The forward Euler method is defined as:

$$\Rightarrow Y_{i+1} = Y_i + h \cdot f(x_i, Y_i)$$

- Computing the method.

$$\Rightarrow \boxed{Y_1} = Y_0 + \frac{1}{10} \cdot f(0, 1) = 1 - \frac{1}{10} \cdot 25 \cdot 1 = \boxed{-1.5}$$

No Stable!

$$\Rightarrow \boxed{Y_2} = Y_1 + \frac{1}{10} \cdot f\left(\frac{1}{10}, -1.5\right) = -1.5 - \frac{1}{10} 25(-1.5)^{3.5} \boxed{\neq \cancel{\ldots 834} \notin \mathbb{R}}$$

indeble

f) The analytical solution is

$$y(x) = \left(\frac{125x + 2}{2} \right)^{-2/5}$$

Using Matlab codes, indicate the maximum stable interval size possible for forward Euler method from the following; $h = 1/10$, $h = 1/15$; $h = 1/30$; $h = 1/45$; $h = 1/90$. How does your choice compare with the stability condition?

- Using MatLab codes, we can see that:
  - $h = 1/10 \Rightarrow$ unstable
  - $h = 1/15 \Rightarrow$ unstable
  - $h = 1/30 \Rightarrow$ ~~un~~stable
  - $h = 1/45 \Rightarrow$ stable
  - $h = 1/90 \Rightarrow$ stable

- So the maximum interval size is $h = 1/30$.
- To compute the stability condition, we have to linearize the function by using Taylor series at $y = 1$

$$\Rightarrow f(y) = -25y^{3.5} \approx f(1) + f'(1) \cdot (y-1)$$

$$\Rightarrow f(y) = -25 - 87{,}5 \cdot (y-1) = 62{,}5 - \underbrace{87{,}5}_{\lambda} y$$

- The Euler method is stable for $0 < \lambda h < 2$ so

$$\Rightarrow \boxed{h <} \frac{2}{\lambda} = \frac{2}{87,5} = \boxed{0.023}$$

- As we can see the values are a little bit differents, but it is due to the fact that we have linearized nearby 1 while the solution at $x = 1$ is $y \approx 0.18$.

-