Numerical Methods for PDE's, Basics

1*. In 1225, Leonardo of Pisa (also known as Fibonacci) was requested to solve a collection of mathematical problems in order to justify his fame and prestige in the court of Federico II. One of the proposed problems can be formulated as the solution of a third degree polynomial equation.

$$f(x) := x^3 + 2x^2 + 10x - 20 = 0 \qquad (1)$$

Note that the solution of cubic equations was a extremely difficult problem in the 13th century. Here iterative methods are considered for the solution of equation (1).

Compute the unique real root of (1) with 4 iterations of Newton's method with the initial approximation $x^0 = \sqrt[3]{20}$ (which is obtained neglecting the monomials with x and x^2 in front of the monomial with x^3). Plot the convergence graphic. Does Newton's method behave as expected?

- First of all, we have to verify that the requirements of the method are satisfied.

$\Rightarrow f'(x) = 3x^2 + 4x + 10$, it exists $\forall x$. ✓

$\rightarrow f'(x) \neq 0 \ \forall x \in \mathbb{R}$ ✓, both solutions are $\mathbb{C}$

- Once we have verified the requirements, we can compute the method:

$$\Rightarrow x^1 = x^0 - \frac{f(x^0)}{f'(x^0)} = \sqrt[3]{20} - \frac{f(\sqrt[3]{20})}{f'(\sqrt[3]{20})} = 1{,}73959$$

$$\Rightarrow x^2 = x^1 - \frac{f(x^1)}{f'(x^1)} = 1{,}40497$$

$$\Rightarrow x^3 = x^2 - \frac{f(x^2)}{f'(x^2)} = 1,36918$$

$$\Rightarrow \boxed{x^4} = x^3 - \frac{f(x^3)}{f'(x^3)} = \boxed{1,36881}$$

- To plot the convergence graphic we have to use the analitical solution (in case we didn't have it, we would use the following step solution for each iteration)

$$\Rightarrow \underline{\alpha = 1.36881}$$

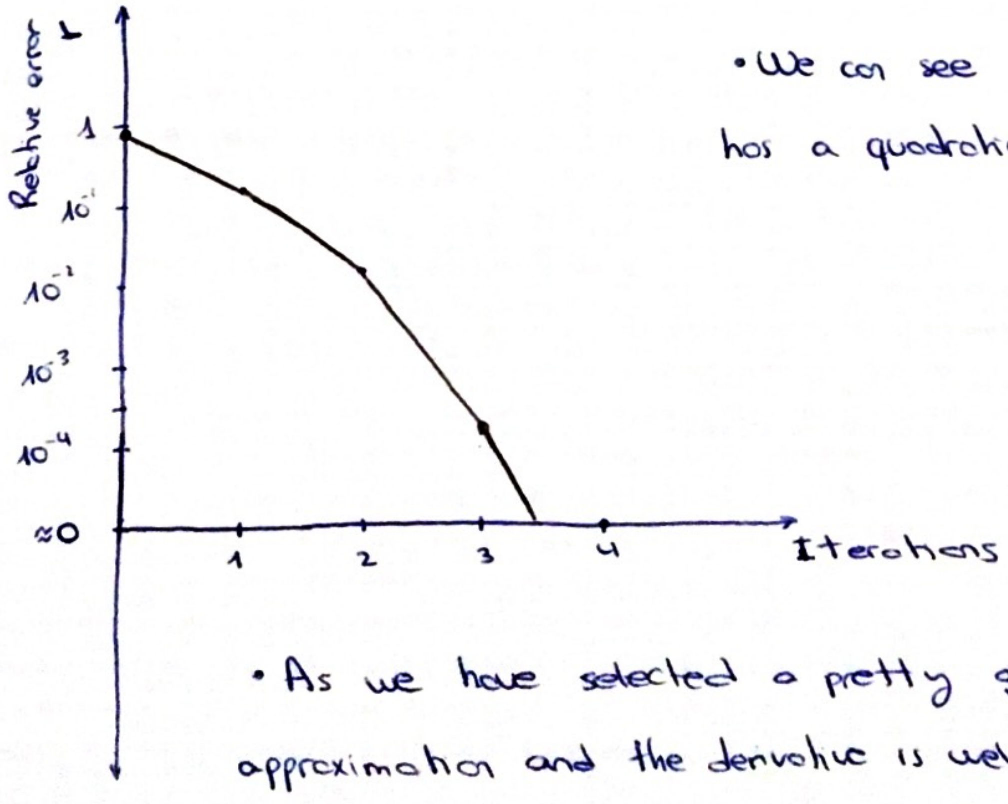
- Let's compute the relative error r_i for each iteration:

$$\Rightarrow r_0 = \left| \frac{x^0 - \alpha}{\alpha} \right| = 0.98305 \quad ; \quad r_1 = \left| \frac{x^1 - \alpha}{\alpha} \right| = 0.27088 ;$$

$$\Rightarrow r_2 = \left| \frac{x^2 - \alpha}{\alpha} \right| = 0.02642 \quad ; \quad r_3 = \left| \frac{x^3 - \alpha}{\alpha} \right| = 2.7031 \cdot 10^{-4}$$

$$\Rightarrow r_4 = \left| \frac{x^4 - \alpha}{\alpha} \right| \approx 0 .$$

- To end we can plot the convergence graphic



- We can see that the method has a quadratic convergence.

- As we have selected a pretty good initial approximation and the derivative is well calculated the method is converging quadratically, as it was expected.

5. We are interested in the definition of third-order quadratures in interval $(0,1)$

a) Determine the minimum number of integration points, and specify the integration points and weights.

- As we are interested in obtaining the minimum number of integration points to integrate exactly a third degree polynomial, we have to use Gauss-quadratures.

- These quadratures guarantee you that with $n+1$ points they are able to integrate exactly a polynomial of degree $2n+1$. In this case:

$$\Rightarrow 2n+1=3 \Leftrightarrow n=1 \longrightarrow n+1=2$$

- $\boxed{\text{So the minimum number of integration points will be 2 points}}$

- Now, we want to obtain the integration points and their weights. We want to integrate in interval $(0,1)$:

$$\Rightarrow I = \int_0^1 F(x)\, dx$$

- We know that the Gauss-Legendre quadrature

$$\longrightarrow \int_{-1}^{1} f(z)\, dz = \sum_{i=0}^{1} w_i f(z_i)$$

has the weights and points tabulated as:

$$\Rightarrow z_0 = \frac{-\sqrt{3}}{3} \;;\; z_1 = \frac{\sqrt{3}}{3} \;;\; w_0 = 1 \;;\; w_1 = 1.$$

- So, we have to do a change of variable from $[-1,1]$ to $[0,1]$.

$$\Rightarrow x = \frac{1}{2}z + \frac{1}{2} \quad ; \quad dx = \frac{1}{2}dz$$

$$\Rightarrow \boxed{I =} \int_0^1 F(x)\,dx = \frac{1}{2}\int_{-1}^{1} F\left(\frac{1}{2}z + \frac{1}{2}\right)\cdot dz = \frac{1}{2}\int_{-1}^{1} f(z)\,dz =$$

$$= \frac{1}{2}\sum_{i=0}^{1} w_i \cdot f(z_i) = \frac{1}{2}\sum_{i=0}^{1} w_i \cdot F\left(\frac{1}{2}z_i + \frac{1}{2}\right) = \boxed{\frac{1}{2}\sum_{i=0}^{1} w_i F(x_i)}$$

- We can conclude that the weights will be the same and the integration points will be x_0 and x_1 which can be found through the change of variable as:

$$\Rightarrow \boxed{x_0} = \frac{1}{2}\cdot\left(\frac{-\sqrt{3}}{3}\right) + \frac{1}{2} = \boxed{\frac{3-\sqrt{3}}{6}}, \quad \boxed{w_0 = 1}$$

$$\boxed{x_1} = \frac{1}{2}\left(\frac{\sqrt{3}}{3}\right) + \frac{1}{2} = \boxed{\frac{3+\sqrt{3}}{6}}, \quad \boxed{w_1 = 1}$$

b) Is it possible to obtain a third-order quadrature with the following four integration points: $x_0 = \frac{1}{4}$, $x_1 = \frac{1}{2}$, $x_2 = \frac{3}{4}$ and $x_3 = 1$? If it is possible, compute the corresponding weights; otherwise, justify whi not.

- First of all we have to realize that the integration points are given so we can not use Gauss quadratures.

- We decide to use Newton-Cotes quadratures, as this quadrature is a third-order quadrature, an odd quadrature. We realize that we will need only 3 points to integrate exactly, so we take the interior points and decide to use the open quadrature of Newton-Cotes with $n=2$, meaning, the open Simpsons quadrature. The weights are tabulated as:

$$\Rightarrow I = \int_0^1 f(x)dx = \frac{4h}{3}\left\{2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right)\right\} + \frac{14h^5}{45}f^{(4)}(\mu)$$

So,

$$\Rightarrow \boxed{\begin{array}{llll} x_0 = \frac{1}{4}; & w_0 = \frac{8h}{3}; & & \\ x_1 = \frac{1}{2}; & w_1 = -\frac{4h}{3}; & h = \frac{1}{4} & ; E = \frac{14h^5}{45}f^{(4)}(\mu) \\ x_2 = \frac{3}{4}; & w_2 = \frac{8h}{3}; & & \mu \in (0,1) \end{array}}$$

6*a) If n+1 points Gaussian quadrature is used for numerical integration state the order of the polynomial that is integrated exactly.

- In these quadratures the integration points are chosen so that with n+1 points we can integrate exactly polynomials of degree <u>2n+1</u>

b) If n=2 is selected for Gaussian quadrature, which (if any) of the following integrals will be integrated exactly?

i) $\int_0^1 \sin x \, dx \Rightarrow$ No!

- In this case the function which must be integrated is not a polynomial, so we can not guarantee that it will be integrated exactly

ii) $\int_0^1 x^3 \, dx \Rightarrow$ Yes!

- As we have explained in a), which n+1 = 3 integration points, we can integrate exactly, polynomials of degree 2n+1 = 5. So we can integrate $\int_0^1 x^3 dx$ exactly.

iii) $\int_0^1 x^4\,dx \Rightarrow$ Yes!

- The same explanation than ii), as this polynomial is degree $4 \leq 5$, we can guarantee that we can integrate it exactly.

iv) $\int_0^1 x^{5.5}\,dx \Rightarrow$ No!

- The problem here is that we know that with fifth-order quadrature we can integrate exactly polynomial of degree ≤ 5 (but it does not integrate exactly polynomials of degree 6. Here we have an intermediate case, but it is clear that we can't guarantee that this integral will be integrate exactly.

7*. Compute $\int_0^1 12x\,dx$, $\int_0^1 (5x^3+2x)\,dx$ by hand calculation using

i) Trapezoidal rule over 2 uniform intervals

ii) Simpson's rule over 2 uniform intervals

Compute the error of both approximations. Are the methods behaving as expected?

- $\int_0^1 12x\,dx$

  - Trapezoidal rule

$$\Rightarrow \boxed{I \doteq} \int_0^1 12x\,dx = \frac{1}{4}\left\{f(0)+2f\left(\frac{1}{2}\right)+f(1)\right\} = \frac{1}{4}\left\{0+2\cdot 6+12\right\} \boxed{=6}$$

  - Simpson's rule

$$\Rightarrow \boxed{I \doteq} \int_0^1 12x\,dx = \frac{1}{12}\left\{f(0)+4f\left(\frac{1}{4}\right)+2f\left(\frac{1}{2}\right)+4f\left(\frac{3}{4}\right)+f(1)\right\} =$$

$$= \frac{1}{12}\left\{12 + 12 + 3\cdot 12 + 12\right\} = \boxed{6}$$

- In this case, the analytical solution is 6 and both methods have integrated it exactly.

- $\int_0^1 (5x^3 + 2x)\,dx$

  - Trapezoidal rule

$$\Rightarrow \boxed{I} = \int_0^1 (5x^3+2x)dx = \frac{1}{4}\left\{f(0) + 2f\left(\tfrac{1}{2}\right) + f(1)\right\} = \frac{1}{4}\left\{\frac{13}{4} + 7\right\} = \boxed{\frac{41}{16}}$$

  - Simpson's rule

$$\Rightarrow \boxed{I} = \int_0^1 (5x^3+2x)dx = \frac{1}{12}\left\{f(0) + 4f\left(\tfrac{1}{4}\right) + 2f\left(\tfrac{1}{2}\right) + 4f\left(\tfrac{3}{4}\right) + f(1)\right\} =$$

$$= \frac{1}{12}\left\{\frac{37}{16} + \frac{13}{4} + \frac{231}{16} + 7\right\} = \boxed{\frac{9}{4}}$$

- In this case, the analytical solution is 9/4, so we have an error using the trapezoidal rule whose value is

$$\Rightarrow \boxed{E = -0.3125}$$

and using Simpson's rule we can integrate it exactly.

- Both methods behaved as expected, with trapezoidal rule we can integrate exactly polynomial of degree ≤ 1, for that reason there is an error when we compute a third-order polynomial integral. Whilst, with the Simpson's rule we can compute exactly polynomial of degree ≤ 3, for that reason there is no error in it.

10*. Perform the numerical integration of

$$\int_0^1 \int_0^1 (9x^3 + 8x^2)\cdot(y^3+y)\,dx\,dy$$

using Simpson's rule in each direction. Is the approximation behaving as expected?

- We will separete the integrate in each direction, so:

$$\Rightarrow \int_0^1 \int_0^1 (9x^3 + 8x^2)\cdot(y^3+y)\,dx\,dy = \int_0^1 (9x^3+8x^2)\,dx \cdot \int_0^1 (y^3+y)\,dy$$

- Let's compute each integral using Simpson's rule:

$$\rightarrow I_1 = \int_0^1 (9x^3+8x^2)\,dx = \frac{1}{6}\left\{f(0)+4f\left(\tfrac{1}{2}\right)+f(1)\right\} = \frac{1}{6}\left\{\frac{25}{2}+17\right\} = \frac{59}{12}$$

$$\rightarrow I_2 = \int_0^1 (y^3+y)\,dy = \frac{1}{6}\left\{f(0)+4f\left(\tfrac{1}{2}\right)+f(1)\right\} = \frac{1}{6}\left\{\frac{5}{2}+2\right\} = \frac{3}{4}$$

So the solution will be:

$$\Rightarrow \boxed{I} = \int_0^1 \int_0^1 (9x^3+8x^2)\cdot(y^3+y)\,dx\,dy = I_1 \cdot I_2 = \boxed{\frac{59}{16}}$$

- In each direction we had third-degree polynomial, so Simpson's rule should be able to integrate it exactly. If the method has behaved as expected, there shouldn't be error.

Let's compute analytically the solution:

$$\Rightarrow I = \int_0^1 (9x^3+8x^2)\,dx \cdot \int_0^1 (y^3+y)\,dy = \left[\frac{9}{4}x^4 + \frac{8}{3}x^3\right]_0^1 \cdot \left[\frac{1}{4}y^4 + \frac{1}{2}y^2\right]_0^1 =$$

$$= \frac{59}{12}\cdot\frac{3}{4} = \frac{59}{16} \Rightarrow \boxed{E=0}$$

So the method has behaved as expected.