5.) Third-order quadrature in (0,1).

a.) at least 3 integration points.

x_i	w_i
0.1127	5/9
0.5000	8/9
0.8873	5/9

b.) I think it's not possible because in an open quadrature (a,b), point a should be smaller than x_0, $a < x_0$, and $b > x_n$. In the case $x_n = x_3 = 1$, $x_3 > b$, therefore it shouldn't be possible.

6.) a.) The rule (2n+1) states it can yield an exact result for polynomials of order $2n-1$ or less, n given number of points, hence:

$$2n-1 = \text{order of polynomial.}$$

$$2(n+1)-1 = \quad ''$$

#of points $\quad 2n+2-1 = ''$

$2n+1$ = order of polynomial that can be solved exactly.

b.) If $n=2$, which can be integrated exactly:

$$2n-1 = 2\cdot(2)-1 = 3. \rightarrow \text{order} \leq 3.$$

Options 1 and 2 can, 3 and 4 can <u>not</u> be solved exactly.

7.) $\int_0^1 12x\,dx$, $\int_0^1 (5x^3+2x)\,dx$ $\longrightarrow \begin{cases} 1.\ \text{Trapezoidal},\ 2 \text{ uniform intervals.} \\ 2.\ \text{Simpson's},\ 2\ \text{"} \quad \text{"}. \end{cases}$

• <u>Trapezoidal</u>

a.) $\int_0^1 12x\,dx \longrightarrow \begin{matrix} m=2 \\ h=1 \end{matrix}$

<u>Interval 1</u> $= [0, 0.5]$

$$I_1 = \int_0^{0.5} 12x\,dx = \frac{1}{4}\left[f(0) + f(1/2)\right] = \frac{1}{4}\left[0+6\right] = \boxed{\frac{6}{4}}$$

<u>Interval 2</u> $= [0.5, 1]$

$$I_2 = \int_{0.5}^1 12x\,dx = \frac{1}{4}\left[f(0.5) + f(1)\right] = \frac{1}{4}\left[6+12\right] = \boxed{\frac{18}{4}}$$

$$I_1 + I_2 = \frac{24}{4} = 6.$$

<u>Complete Interval</u> $= [0,1]$

$$I_T = \int_0^1 12x\,dx = \frac{12}{2} = \boxed{6}.$$

• <u>Simpson's</u> $\begin{matrix} m=2 \\ h=2 \end{matrix}$

<u>Interval 1</u> $= [0, 0.5] \longrightarrow x_0 = 0,\ x_1 = 1/4,\ x_2 = 1/2.$

$$I_1 = \int_0^{1/2} 12x\,dx = \frac{1}{12}\left[f(0) + 4f(1/4) + f(1/2)\right] = \frac{1}{12}\left[0+12+6\right] = \boxed{\frac{18}{12}}$$

<u>Interval 2</u> $= [0.5, 1] \longrightarrow x_0 = 1/2,\ x_1 = 3/4,\ x_2 = 1.$

$$I_2 = \int_{1/2}^1 12x\,dx = \frac{1}{12}\left[f(1/2) + 4f(3/4) + f(1)\right] = \frac{1}{12}\left[6+36+12\right] = \boxed{\frac{54}{12}}$$

$$I_1 + I_2 = \frac{18+54}{12} = \boxed{6}.$$

<u>Complete Interval</u> $= \int_0^1 12x\,dx = \frac{18}{12} + \frac{54}{12} = \boxed{6}.$

7.) b.) $\int_0^1 (5x^3+2x)\,dx$

• **Trapezoidal** → $m=2$, $h=1$

Interval 1 = $[0, 0.5]$

$$I_1 = \int_0^{0.5} 5x^3+2x\,dx = \frac{1}{4}\left[f(0)+f(\tfrac{1}{2})\right] = \frac{1}{4}\left[0+\frac{13}{8}\right] = \boxed{\frac{13}{32}}$$

Interval 2 = $[0.5, 1]$

$$I_2 = \int_{0.5}^1 5x^3+2x\,dx = \frac{1}{4}\left[f(\tfrac{1}{2})+f(1)\right] = \frac{1}{4}\left[\frac{13}{8}+7\right] = \frac{13}{32}+\frac{56}{32} = \boxed{\frac{69}{32}}$$

Complete Interval = $I_T = \int_0^1 (5x^3+2x)\,dx = \dfrac{13+69}{32} = \boxed{\dfrac{82}{32}}$.

• **Simpson's**

Interval 1 = $[0, 0.5]$ → $x_0=0$, $x_1=1/4$, $x_2=1/2$.

$$I_1 = \int_0^{1/2} 5x^3+2x\,dx = \frac{1}{12}\left[f(0)+4f(\tfrac{1}{4})+f(\tfrac{1}{2})\right] = \frac{1}{12}\left[0+4\cdot\frac{37}{64}+\frac{13}{8}\right] = \boxed{\frac{21}{64}}$$

Interval 2 = $[0.5, 1]$ → $x_0=1/2$, $x_1=3/4$, $x_2=1$.

$$I_2 = \int_{1/2}^1 5x^3+2x\,dx = \frac{1}{12}\left[f(\tfrac{1}{2})+4f(\tfrac{3}{4})+f(1)\right] = \frac{1}{12}\left[\frac{13}{8}+\frac{231}{16}+7\right] = \frac{123}{64}.$$

Complete Interval = $\dfrac{21}{64}+\dfrac{123}{64} = \boxed{\dfrac{144}{64}} \rightarrow \boxed{2.25}$

$\dfrac{72}{32}$.

Exact Solution = $\int_0^1 5x^3+2x\,dx = \frac{5}{4}x^4+x^2\Big|_0^1 = \frac{9}{4} = \boxed{2.25}$

10.) Numerical Integration of:

$$\int_0^1 \int_0^1 (9x^3 + 8x^2)(y^3 + y)\, dx\, dy$$

Simpson's , $\underline{n=2}$ $\longrightarrow$ $x_0 = 0$ $\quad x_1 = 1/2$ $\quad x_2 = 1$

$y_0 = 0$ $\quad y_1 = 1/2$ $\quad y_2 = 1$

$\underline{\text{First}} \rightarrow y = y_0 = 0.$

$$I_1 = \int_0^1 f(x, 0)\, dx = \frac{1}{6}\left[f(0,0) + 4f(1/2, 0) + f(1, 0)\right] = 0.$$

$\underline{\text{Second}} \rightarrow y = y_1 = 1/2$

$$I_2 = \int_0^1 f(x, 1/2)\, dx = \frac{1}{6}\left[f(0, 1/2) + 4f(1/2, 1/2) + f(1, 1/2)\right] = \boxed{\frac{265}{96}}.$$

$\underline{\text{Third}} \rightarrow y = y_2 = 1$

$$I_3 = \int_0^1 f(x, 1)\, dx = \frac{1}{6}\left[f(0,1) + 4f(1/2, 1) + f(1,1)\right] = \frac{1}{6}\left[0 + \frac{425}{4} + 34\right] = \boxed{\frac{56}{6}}$$

$\underline{\text{Y-direction}}$

$$I_T = \frac{1}{6}\left[I_1 + 4I_2 + I_3\right] = \boxed{\frac{163}{46}} \approx \boxed{3.396}.$$

$\underline{\text{Calculated Error}} \approx \boxed{0.078}.$