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Homework 3

Solution 2 Given PDE: $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \quad x \in (0,1)$

Initial condition: $u(x,0) = \sin(2\pi x)$

Periodic Boundary Condition: $u(0,t) = u(1,t)$

(a) $$\frac{\partial u}{\partial t} = \frac{u_i^{n+1} - u_i^n}{\Delta t} + O(\Delta t)$$

$$\frac{\partial u}{\partial x} = \frac{u_{i+1}^{n+1} - u_i^{n+1}}{\Delta x} + O(\Delta x)$$

Neglecting truncation error and putting values in the given equation we obtain

$$u_i^{n+1} - u_i^n = \frac{a\Delta t}{\Delta x}\left(u_i^{n+1} - u_{i+1}^{n+1}\right)$$

$r = \frac{a\Delta t}{\Delta x}$ thus; $u_i^{n+1} - u_i^n = r\left(u_i^{n+1} - u_{i+1}^{n+1}\right)$

$$(1-r)\, u_i^{(n+1)} + r\, u_{i+1}^{n+1} = u_i^n$$

(b) Assuming there are $(m+1)$ nodes from 0 to m

$$\left.\begin{aligned} (1-r)u_0^{n+1} + r u_1^{n+1} &= u_0^n \\ (1-r)u_1^{n+1} + r u_2^{n+1} &= u_1^n \\ &\vdots \\ (1-r)u_{m-1}^{n+1} + r u_{m-1}^{n+1} &= u_{m-1}^n \end{aligned}\right\}$$

Thus we have $(m+1)$ equations and $(m+1)$ unknowns at each time step.
The periodic BC is treated with other m equations as a system of $(m+1)$ linear equations.

(c) Direct Method → Gauss Elimination
Indirect Method → Gauss Seidel method

(d) Fill in Matrix

$$\begin{bmatrix} (1-r) & r & 0 & - & \cdots & - \\ 0 & (1-r) & r & 0 & - & - \\ | & 0 & (1-r) & r & 0 & - \\ | & & \ddots & \ddots & & \\ | & & & \ddots & \ddots & \\ & & & & (1-r) & r \\ r & - & - & - & - & (1-r) \end{bmatrix}$$

Solution 4 (a) $\frac{\partial u}{\partial t} = \frac{u_i^{n+1} - u_i^n}{\Delta t}$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{(\Delta x)^2}$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\nu(u_{i-1}^n - 2u_i^n + u_{i+1}^n)}{(\Delta x)^2} + \sigma u_i^n$$

$$u_i^{n+1} = r(u_{i-1}^n - 2u_i^n + u_{i+1}^n) + \sigma u_i^n \Delta t + u_i^n$$

$$u_i^{n+1} = r u_{i-1}^n + (1 + \sigma\Delta t - 2r) u_i^n + r u_{i+1}^n$$

(b) For $\sigma = 0$

$$u_t = \nu u_{xx}$$

So FTCS scheme would be

$$u_i^{n+1} = r u_{i-1} + (1-2r) u_i^n + r u_{i+1}^n$$

For $\nu = 0 \Rightarrow r = 0$

so scheme would be

$$u_i^{n+1} = (1 + \sigma\Delta t) u_i^n$$

(c) $\Delta x = 0.25$

The initial condition is given so, we know initial u values:

$$u_0^0 = 0; \quad u_1^0 = 0, \quad u_2^0 = 1, \quad u_3^0 = 0, \quad u_4^0 = 0$$

$$r = \frac{\nu\Delta t}{\Delta x^2} = 0.16 \quad ; \quad \sigma\Delta t = -0.01$$

At 1st time step

$$u_0^1 = 0; \quad u_1^1 = r u_0^0 + (1 + \sigma\Delta t - 2r) u_1^0 + r u_2^0$$
$$= 0.16$$

Dirichlet BC at $x = 0$

$$x(0,t) = 0 \Rightarrow u_0^n = 0 \quad \forall n$$

Neumann BC at $x = 1 \Rightarrow u_x(1,t) = 0$

Applying central difference for derivative at m.

$$\left.\frac{\partial u}{\partial x}\right|_m^n = \frac{u_{m+1} - u_{m-1}}{2\Delta x} = 0$$

$u_{m+1} = u_{m-1}$ from neumann BC

Substituting in FTCS scheme at node "n"

$$u_m^{n+1} = r u_{m-1}^{n} + (1+\sigma\Delta t - 2r) u_m^n$$

Therefore we can find the u values at boundary $x=1$

(c) Similarly we have

$u_2^1 = 0.67$; $u_3^1 = 0.16$

$u_4^1 = 2r u_3^0 + (1+\sigma\Delta t - 2r) u_4^0 = 0$

For 2nd time step

$u_0^2 = 0$; $u_1^2 = 0.2144$, $u_2^2 = 0.5001$, $u_3^2 = 0.2144$, $u_4^2 = 0.0512$

(d) Implicit FD scheme

Let us choose Backward in time, central in space (BTCS)

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \nu \frac{\left(u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}\right)}{(\Delta x)^2} + \sigma u_i^{n+1}$$

$$u_i^{n+1} - u_i^n = r(u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}) + \sigma\Delta t\, u_i^{n+1}$$

where $r = \frac{\nu\Delta t}{\Delta x^2}$

$$-r u_{i-1}^{n+1} + (1+2r-\sigma\Delta t) u_i^{n+1} - r u_{i+1}^{n+1} = u_i^n$$

with same initial condⁿ and BC

$u_0^{n+1} = 0$ and $u_{m+1}^{n+1} = u_{m-1}^{n+1}$ (Using fictitious node for neumann BC at n)

For $i=1$

m equations
$$\left\{\begin{aligned}
&-r u_0^{n+1} + (1+2r-\sigma\Delta t) u_1^{n+1} - r u_2^{n+1} = u_1^2 \\
&-r u_1^{n+1} + (1+2r-\sigma\Delta t) u_2^{n+1} - r u_3^{n+1} = u_2^n \\
&\quad\vdots \\
&-r u_{m-2}^{n+1} + (1+2r-\sigma\Delta t) u_{m-1}^{n+1} - r u_m^{n+1} = u_{m-1}^n \\
&-2r u_{m-1}^{n+1} + (1+2r-\sigma\Delta t) u_m^{n+1} = u_m^n
\end{aligned}\right.$$

The matrix of linear system of equations

$$
\begin{bmatrix}
(1+2r-\sigma\Delta t) & -r & 0 & \cdots & \cdots & 0 \\
-r & (1+2r-\sigma\Delta t) & -r & & & \\
0 & \ddots & \ddots & \ddots & & \\
\vdots & & \ddots & \ddots & \ddots & \\
0 & 0 & -r & & (1+2r-\sigma\Delta t) & -r \\
0 & \cdots & 0 & & -2r & (1+2r-\sigma\Delta t)
\end{bmatrix}
$$