

1-
 $f(x) = x^3 + 2x^2 + 10x - 20 = 0$

$f'(x) = 3x^2 + 4x + 10$

$x^{k+1} = x^k - \frac{f(x^k)}{f'(x^k)}$

$x^0 = \sqrt[3]{20}$

$x^1 = 1,73959$

$x^2 = 1,40497$

$x^3 = 1,36918$

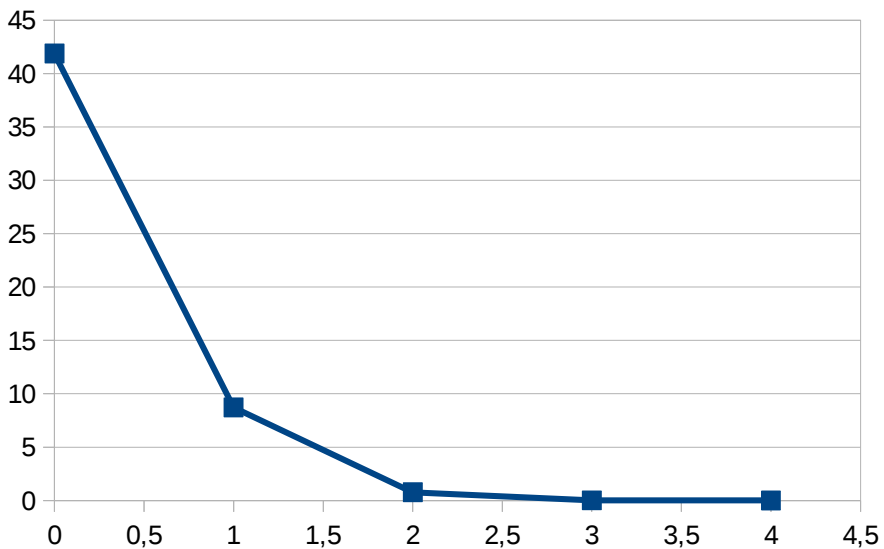
$x^4 = 1,36881$

Exact solution $\approx 1,388$

5- Analytic

a) A 3rd order quadrature will need a minimum of 2 points if using a Gauss quadrature:

$3 = 2n + 1 \Rightarrow n = 1 \Rightarrow z_0$ and z_1 are the points needed



$$\int_0^1 1 dx = 1 = w_0 + w_1$$

$$\int_0^1 x dx = \frac{1}{2} = w_0 z_0 + w_1 z_1$$

$$\int_0^1 x^2 dx = \frac{1}{3} = w_0 z_0^2 + w_1 z_1^2$$

$$\int_0^1 x^3 dx = \frac{1}{4} = w_0 z_0^3 + w_1 z_1^3$$

Solving the system:

$$w_0 = \frac{1}{2}$$

$$w_1 = \frac{1}{2}$$

$$z_0 = \frac{1}{6}(3 + \sqrt{3})$$

$$z_1 = \frac{1}{6}(3 - \sqrt{3})$$

b) Given the 4 equally spaced points it is possible to obtain a 3rd order quadrature because Newton-Cotes only requires 3 points

$$\int_0^1 1 dx = 1 = w_0 + w_1 + w_2 + w_3$$

$$\int_0^1 x dx = \frac{1}{2} = \frac{w_0}{4} + \frac{w_1}{2} + \frac{3w_2}{4} + w_3$$

$$\int_0^1 x^2 dx = \frac{1}{3} = \frac{w_0}{16} + \frac{w_1}{4} + \frac{9w_2}{4} + w_3$$

$$\int_0^1 x^3 dx = \frac{1}{4} = \frac{w_0}{64} + \frac{w_1}{8} + \frac{27w_2}{64} + w_3$$

Solving the system:

$$w_0 = \frac{2}{3}$$

$$w_1 = -\frac{1}{3}$$

$$w_2 = \frac{2}{3}$$

$$w_3 = 0 \rightarrow \text{the 4th point is not required}$$

6.

a) $n+1$ points that go from z_0 to $z_n \Rightarrow$ Order of the polynomial integrated exactly will be $2n+1$

b) $n=2 \Rightarrow$ Order of the polynomials integrated exactly $\leq 2n+1=5$

i) $\int_0^1 \sin x \, dx \rightarrow$ Can't be ^{computed} written exactly because the Taylor series only will be integrated exactly until order 5; the next terms will have an error

ii) $\int_0^1 x^3 \, dx \rightarrow$ Order $< 5 \Rightarrow$ It can be computed exactly

iii) $\int_0^1 x^4 \, dx \rightarrow$ Order $< 4 \Rightarrow$ It can be computed exactly

iv) $\int_0^1 x^{5.5} \, dx = \int_0^1 x^5 \, dx - \int_0^1 x \, dx \Rightarrow$ It can't be computed exactly

7.

$$I_1 = \int_0^1 12x \, dx, \quad I_2 = \int_0^1 (5x^3 + 2x) \, dx$$

a) Trapezoidal rule over 2 intervals

$$\int_a^b f(x) \, dx \approx \frac{h}{2} [f(a) + 2f(a+h) + f(b)] \quad \text{with } \begin{cases} a=0 \\ h=1/2 \\ b=1 \end{cases}$$

$$I_1 \approx \frac{12}{4} [0 + 1 + 1] = 6; \quad \text{Err}_1 = 0$$

$$I_2 \approx \frac{1}{4} [0 + 2 \cdot \frac{13}{8} + 7] = 2,5625, \quad \text{Err}_2 = 0,3125$$

The trapezoidal rule integrates exactly polynomials 1st order - polynomials

b) Simpsons rule over 2 intervals

$$\int_a^b f(x) \, dx \approx \frac{h}{3} [f(a) + 4f(a+h) + f(b)] \quad \text{with } \begin{cases} a=0 \\ h=1/2 \\ b=1 \end{cases}$$

$$I_1 \approx \frac{12}{6} [0 + 2 + 1] = 6; \quad \text{Err}_1 = 0$$

$$I_2 \approx \frac{1}{6} [0 + \frac{13}{2} + 7] = 2,25; \quad \text{Err}_2 = 0$$

Simpsons rule integrates exactly up to 3rd order polynomials

10-

$$\int_0^1 (9x^3 + 8x^2) \, dx = I_1 \quad \int_0^1 (y^3 + y) \, dy = I_2$$

$$\text{Simpsons rule: } \int_a^b f(x) \, dx \approx \frac{h}{3} [f(a) + 4f(a+h) + f(b)] \quad \text{with } \begin{cases} a=0 \\ h=1/2 \\ b=1 \end{cases}$$

$$I_1 \approx 4,91666 \quad \left\{ \begin{array}{l} I = I_1 \cdot I_2 = 3,6875, \quad \text{Err}_2 = 0 \Rightarrow \text{The integral is exact because} \\ I_2 \approx \frac{3}{4} \end{array} \right.$$

in each direction the polynomials can be integrated exactly.