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Numerical Methods for Partial differential Equations

Exercises: Basics

1. In 1225, Leonardo of Pisa (also known as Fibonacci) was requested to solve a collection of mathematical problems in order to justify his fame and prestige in the court of Federico II. One of the proposed problems can be formulated as the solution of a third degree polynomial equation

$$f(x) = x^3 + 2x^2 + 10x - 20 = 0 \qquad (1)$$

Note that the solution of cubic equations was a extremely difficult problem in the 13th century. Here iterative methods are considered for the solution of the equation.
Compute the unique real root of (1) with 4 iterations of Newton's Method with the initial approximation $x_0 = \sqrt[3]{20}$ (which is obtained neglecting the monomials with $x$ and $x^2$ in front of the monomials with $x^3$). Plot the convergence graphic. Does Newton's method behave as expected?

We are going to use Newton's method. Therefore:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$ We need the derivative of our function. $\rightarrow$ $f'(x) = 3x^2 + 4x + 10$

$\Rightarrow$ Iteration 1:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \sqrt[3]{20} - \frac{(\sqrt[3]{20})^3 + 2(\sqrt[3]{20})^2 + 10(\sqrt[3]{20}) - 20}{3(\sqrt[3]{20})^2 + 4(\sqrt[3]{20}) + 10} = 1.739$$

$\Rightarrow$ Iteration 2:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.739 - 0.334 = 1.405$$

$\Rightarrow$ Iteration 3:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.405 - 0.035 = 1.37$$

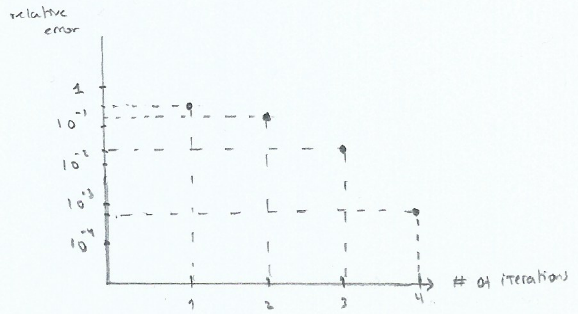
$\Rightarrow$ Iteration 4:

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_2)} = 1.37 - 0.001 = 1.369$$

Now we are going to plot the convergence graphic. In order to do so, we need to compute the relative error.

$$\text{rel. error} = \frac{x_n - x_{n+1}}{x_{n+1}}$$

$\text{rel. error}_1 = \frac{x_0 - x_1}{x_1} = 0.56$ $\quad$ $\text{rel. error}_2 = \frac{x_1 - x_2}{x_2} = 0.237$ $\quad$ $\text{rel. error}_3 = \frac{x_2 - x_3}{x_3} = 0.025$ $\quad$ $\text{rel. error}_4 = \frac{x_3 - x_4}{x_4} = 0.0007$




We can see that the method behaves as expected, since the rate of convergence is quadratic. This way, we can affirm that our initial guess was a good one.

5. We are interested in the definition of third-order numerical quadratures in the interval $(0,1)$.

a) Determine the minimum number of integration points, and specify the integration points and weights.

B) Is it possible to obtain a third order quadrature with the following four integration points: $x_0 = 1/4$, $x_1 = 1/2$, $x_2 = 3/4$ and $x_3 = 1$? If it is possible, compute the corresponding weights, otherwise, justify why not.

a) Third order quadratures integrate perfectly polynomials of degree $\leq 3$.

Using Gauss quadratures, with $n=1$ (2 points), we can integrate exactly polynomials of order $2n+1$. With $n=1$, we get an order of 3. This way, the minimum number of integration points is 2.

Now we have to compute this points and their corresponding weights.

For Gauss-Legendre quadratures and $n=1$ (order 3):

$$\begin{cases} z_0 = -\sqrt{3}/3 & z_1 = \sqrt{3}/3 \\ w_0 = 1 & w_1 = 1 \end{cases} \qquad \int_{-1}^{1} f(z)\,dz \simeq \sum_{i=0}^{n} w_i f(z_i) \qquad I = \int_a^b F(x)\,dx$$

Our interval is between 0 and 1. This way, we need to make a change of variable:

$$x = \frac{b-a}{2} z + \frac{a+b}{2}, \qquad dx = \frac{b-a}{2}\,dz \quad \text{thus:}$$

$$I = \frac{b-a}{2} \int_{-1}^{1} F\left(\frac{b-a}{2} z + \frac{a+b}{2}\right) dz = \frac{b-a}{2} \int_{-1}^{1} f(z)\,dz$$

Using Gauss-Legendre quadratures:

$$I \simeq \frac{b-a}{2} \sum_{i=0}^{n} w_i f(z_i)$$

Consequently:

$$I = \frac{b-a}{2} \sum_{i=0}^{n} w_i F\left(\frac{b-a}{2} z_i + \frac{a+b}{2}\right) + E_n$$

For our problem, $a=0$ and $b=1$:

$$\boxed{\begin{array}{ll} x_0 = \frac{1}{2} \cdot \left(\frac{-\sqrt{3}}{3}\right) + \frac{1}{2} = \frac{3-\sqrt{3}}{6} & w_0 = 1 \\ x_1 = \frac{1}{2} \frac{\sqrt{3}}{3} + \frac{1}{2} = \frac{3+\sqrt{3}}{6} & w_1 = 1 \end{array}} \quad \rightarrow \quad I = \frac{1}{2} \sum_{i=0}^{1} w_i F\left(\frac{1}{2} z_i + \frac{1}{2}\right) + E_n$$

B) Is it possible to obtain a third-order quadrature with the following four integration points: $x_0 = 1/4$, $x_1 = 1/2$, $x_2 = 3/4$, and $x_3 = 1$? If it is possible, compute the corresponding weights, otherwise, justify why not.

Since the integration points are given, we can't use Gauss quadratures. We can nevertheless use Newton-Cotes quadratures.

We want to obtain a third-order quadrature (an odd one. We only need 3 points to integrate exactly, so we'll take $x_0$, $x_1$ and $x_2$ and use a Newton-Cotes quadrature with $n=2$ (open Simpson).

Newton-Cotes open formula for $n=2$:

$$I = \frac{4h}{3}\left[2f(x_0) - f(x_1) + 2f(x_2)\right] + \frac{14h^5}{45} f^{4}(\eta) \xrightarrow{} 0$$

$$I = \frac{4h}{3}\left[2f(1/4) - f(1/2) + 2f(3/4)\right]$$ with

$x_0 = 1/4$ $\quad w_0 = \frac{8h}{3}$

$x_1 = 1/2$ $\quad w_1 = -\frac{4h}{3}$

$x_2 = 3/4$ $\quad w_2 = \frac{8h}{3}$

$h = 1/4$

6.

a) If $n+1$ points Gaussian quadrature is used for numerical integration state the order of the polynomial that is integrated exactly.

If we use $n+1$ points, we can integrate exactly polynomials of order $2n+1$, as long as the integration points are NOT predetermined and f can be evaluated in said points.

B) If $n=2$ is selected for Gaussian quadrature, which (if any) of the following integrals will be integrated exactly?

i) $\int_0^1 \sin x \, dx$

ii) $\int_0^1 x^3 dx$

iii) $\int_0^1 x^4$

iv) $\int_0^1 x^{5.5} dx$

$n=2 \Rightarrow$ Order 5, we can integrate exactly polynomials of order $\leq 5$.

i) We can't integrate $\int_0^1 \sin x \, dx$, since this function is not a polynomial.

ii) We can integrate this function, since it is of order 3, which is smaller than 5.

iii) we can integrate this function, since it is of order 4, which is smaller than 5.

iv) we can't integrate $\int_0^1 x^{5.5}$, since its order is higher than 5.

7. Compute $\int_0^1 12x \, dx$, $\int_0^1 (5x^3+2x)dx$ by hand calculation using

i) Trapezoidal rule over 2 uniform intervals

ii) Simpson's rule over 2 uniform intervals.

i) $a=0 \quad b=1$

Trapezoidal composite formula: (equally spaced points)

$$I = \int_a^b f(x)\,dx = \frac{h}{2}\left(f(x_0) + 2\sum_{i=1}^{m-1} f(x_i) + f(x_m)\right) + E_m^T$$ with $h = \frac{b-a}{m}$ being m the number of intervals

In our case:

$h = \frac{1}{2}$ $\quad x_0 = 0$, $x_1 = 1/2$, $x_2 = 1 = x_m$

$\Rightarrow$ For $\int_0^1 12x \, dx$:

$$\int_0^1 12x\,dx = \frac{1}{4}\left(0 + 2\cdot 6 + 12\right) = \frac{24}{4} = 6 \quad \Rightarrow \boxed{\int_0^1 12x\,dx = 6}$$

If we compute this integral analytically, we can check that our result is correct:

$$\int_0^1 12x\,dx = \left.\frac{12x^2}{2}\right]_0^1 = 6 \quad ✓$$ In this case, our relative error would be 0.

$\Rightarrow$ For $\int_0^1 5x^3+2x \, dx$:

$$\int_0^1 5x^3+2x\,dx = \frac{1}{4}\left(0 + 2\cdot\frac{13}{8} + 7\right) = \frac{41}{16} \rightarrow \boxed{\int_0^1 5x^3+2x\,dx = \frac{41}{16}}$$

Now we are going to compute the analytical solution:

$$\int_0^1 5x^3+2x = \left[\frac{5x^4}{4} + \frac{2x^2}{2}\right]_0^1 = \frac{5}{4} + 1 = 9/4$$

We can now compute the relative error, since we are approximating a polynomial of order 3 with straight lines.

$$\text{rel. error} = \left|\frac{I_{analytical} - I_{approx}}{I_{analytical}}\right| = \left|\frac{9/4 - 41/16}{9/4}\right| = 0.138$$

ii) Simpson composite formula: (equally spaced points)

$$I = \int_a^b f(x)\,dx = \frac{h}{3} \sum_{i=1}^{m} \left( f(x_{2i-2}) + 4f(x_{2i-1}) + f(x_{2i}) \right) + E_n^S \quad \text{with } h = \frac{b-a}{2m} \quad \text{in our case } h = \frac{1}{4} \begin{cases} x_0 = 0 \\ x_1 = 1/4 \\ x_2 = 1/2 \\ x_3 = 3/4 \\ x_4 = 1 \end{cases}$$

$\Rightarrow$ For $\int_0^1 12x\,dx$ :

$$I = \int_0^1 12x\,dx = \frac{1}{12} \left( f(x_0) + 4f(x_1) + f(x_2) + f(x_2) + 4f(x_3) + f(x_4) \right) = \frac{1}{12}(0 + 4\cdot 3 + 6 + 6 + 4\cdot 9 + 12) = \frac{72}{12} = 6$$

$$\boxed{\int_0^1 12x\,dx = 6}$$

we can see again that in this case the integral is approximated perfectly, in this case, the relative error is 0 again.

$\Rightarrow$ for $\int_0^1 5x^3 + 2x\,dx$ :

$$I = \int_0^1 5x^3 + 2x\,dx = \frac{1}{12}\left( f(0) + 4f(1/4) + 2f(1/2) + 4f(3/4) + f(1) \right) = \frac{9}{4} \quad \rightarrow \boxed{\int_0^1 5x^3 + 2x\,dx = 9/4}$$

we can see that the result obtained is the same as we have obtained analytically. This way the relative error in this case will also be 0.

10. Perform the numerical integration of

$$\int_0^1 \int_0^1 (9x^3 + 8x^2)(y^3 + y)\,dx\,dy$$

using Simpson's rule in each direction. Is the approximation behaving as expected?

we can re-write this integral in the following way:

$$\int_0^1 9x^3 + 8x^2\,dx \cdot \int_0^1 y^3 + y\,dy \qquad 2 \text{ intervals} \rightarrow \quad h = \frac{b-a}{2m} = \frac{1}{4} \rightarrow \begin{cases} x_0 = 0 \\ x_1 = 1/4 \\ x_2 = 1/2 \\ x_3 = 3/4 \\ x_4 = 1 \end{cases}$$

Let's compute each one of them using Simpson's rule: (the formulas needed have been written in the previous exercises)

$\Rightarrow$ For $\int_0^1 9x^3 + 8x^2\,dx$ :

$$I = \int_0^1 9x^3 + 8x^2 = \frac{1}{12}\left( f(0) + 4f(1/4) + 2f(1/2) + 4f(3/4) + f(1) \right) = \frac{59}{12}$$

Let's compute the analytical solution:

$$\int_0^1 9x^3 + 8x^2 = \left[ \frac{9x^4}{4} + \frac{8x^3}{3} \right]_0^1 = \frac{59}{12}$$

we can see that the result obtained with Simpson's rule is the same as the analytical one. This way, the relative error is 0.

$\Rightarrow$ for $\int_0^1 y^3 + y\,dy = \frac{1}{12}\left( f(0) + 4f(1/4) + 2f(1/2) + 4f(3/4) + f(1) \right) = 3/4$

Let's compute the analytical solution:

$$\int_0^1 y^3 + y\,dy = \left[ \frac{y^4}{4} + \frac{y^2}{2} \right]_0^1 = 3/4$$

we can once again see that the result obtained with Simpson's rule is the same as the analytical one. Therefore we have again a relative error of 0.

Therefore:

$$\boxed{\int_0^1\int_0^1 (9x^3 + 8x^2)(y^3 + y)\,dx\,dy = \int_0^1 9x^3 + 8x^2\,dx \cdot \int_0^1 y^3 + y\,dy = \frac{59}{12} \cdot \frac{3}{4} = \frac{59}{16}}$$