Numerical Methods for Partial Differential Equations

Homework- Basics

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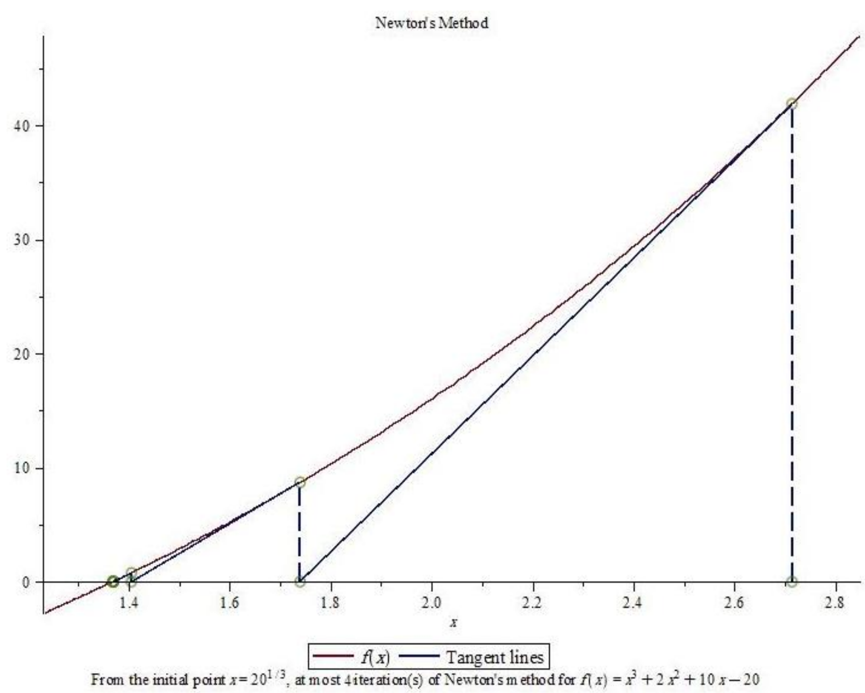
Exercise-1

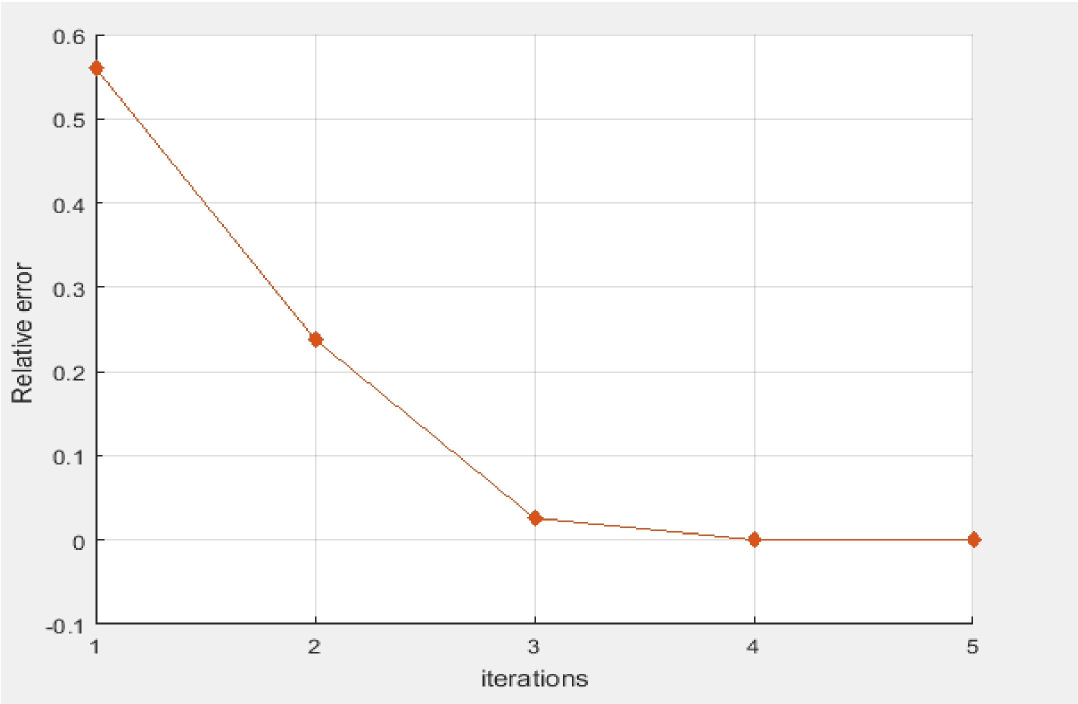
Firstly, Newton method was implanted, as shown in the following Matlab function:

```
x0=20^(1/3); % initial value
f1=@(x) x^3+2*x^2+10*x-20;
df1=@(x) 3*x^2 + 4*x + 10;

maxit=5;
itor=1;
while(1)
    x=x0-f1(x0)/df1(x0);
    ert(itor)=abs ((x-x0)/x);
    y=f1(x);

    hold on
     if (itor>=maxit),break,end
     itor = itor+1;
    x0=x;

end

iteration=[1:1:itor];
plot(iteration,ert);

grid on
```



then with four iterations, the root of the equation and the relative error were as following:

x = 1.3688   relitive_errorr =   2.9738e-08



The above figure represents, the convergence in x, where it can be seen that newton method with given initial value was very effective and worked exactly as it was expected.

Exercise-5

We are interested in the definition of third-order numerical quadrature in interval (0; 1)

a) Determine the minimum number of integration points, and specify the integration points and weights.

Solution –a

The best choice to define a third-order quadrature with minimum number of points, is using Gauss quadratures.

The error in Gauss: $$E_n = \Omega_n f^{2n+2)}(\mu)$$

$$2n + 2 = 4 \rightarrow n = 1$$

Since $n = 1$, the number of the required points is two x_0 & x_1.

In order to specify the integration points and weights, we use Gauss-Legendre quadratures, with n=1 (order = 3)

$$\int_{-1}^{1} f(z)\,dz \;=\; \sum_{i=0}^{n=1} w_i\, f(z_i)$$

$$\int_{-1}^{1} f(z)\,dz \;=\; w_0\, f(z_0) + w_1\, f(z_1)$$

Using:

$$P(z)_0 = 1 \quad P(z)_1 = z \quad P(z)_2 = z^2 \quad P(z)_3 = z^3$$

We get:

$$w_0 + w_1 = 2$$
$$w_0 z_0 + w_1 z_1 = 0$$
$$w_0 z_0^2 + w_1 z_1^2 = \frac{2}{3}$$
$$w_0 z_0^3 + w_1 z_1^3 = 0$$

$$w_0 = 1 \quad w_1 = 1$$

$$z_0 = \frac{1}{\sqrt{3}} \qquad z_1 = -\frac{1}{\sqrt{3}}$$

And now for the interval (0,1), the integration points and weights will be:

$$x_0 = \frac{1}{2} z_0 + \frac{1}{2} = \frac{1+\sqrt{3}}{2\sqrt{3}} \qquad\qquad x_1 = \frac{1}{2} z_1 + \frac{1}{2} = \frac{1-\sqrt{3}}{2\sqrt{3}}$$

$$w_0 = 1 \quad w_1 = 1$$

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b) Is it possible to obtain a third-order quadrature with the following four integration points: $x_0 = 1/4$, $x_1 = 1/2$, $x_2 = 3/4$ and $x_3 = 1$? If it is possible, compute the corresponding weights; otherwise, justify why not.

Solution:

Yes, it is possible using Simpson rule with n=3. With $h = \frac{1}{4}$

$$\text{n=3:} \quad I = \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right] - \frac{3h^5}{80} f^{4)}(\mu)$$

$$w_0 = \frac{3}{32} \qquad w_1 = \frac{9}{32} \qquad w_2 = \frac{9}{32} \qquad w_3 = \frac{3}{32}$$

Exercise-6

a) If n + 1 points Gaussian quadrature is used for numerical integration state the order of the polynomial that is integrated exactly.

Solution-a:

The error in Gaussian quadrature:

$$E_n = \Omega_n f^{2n+2)}(\mu)$$

Since we have $n + 1$ point the order of the error will $2n + 4$ and we will be able to integrate polynomials with order up to $2n + 3$.

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b) Using Gaussian quadrature with n=2, we will be able to integrate exactly only:

$$\int_0^1 x^3 dx \quad \& \quad \int_0^1 x^4 dx$$

Exercise-7

For the first integral, Both methods have obtained the exact value of the integration, which is an expected result since we integrating a first order polynomial, and the used rules are order one (trapezoidal) and three (Simpson).

In the secound integral, Trapezoidal methods had error = 0.13888, while Simpson have obtained the exact value, which is an expected result, since the integral has a third degree polynomial, and the Trapezoidal method of order one and Simpson of order four.

Exercise-10

For the this exercise, the result was quite unusual, because we used a third order quadrature (Simpson) in order to approximate an integral with a sixth-degree polynomial, and the error was only (error= 0.07837), actually it was expected to be higher than this value.

(1) $\int_0^1 12x\,dx$

a) using Trapezoidal rule over 2 uniform interval

$m = 2 \quad n = 1$

- first interval. $[0, \frac{1}{2}]$

$I = \int_0^{1/2} 12x\,dx = \frac{1}{4}[f(0) + f(\frac{1}{2})]$

$= \frac{1}{4}[0 + 6] = \frac{6}{4}$

- Second interval $[\frac{1}{2}, 1]$

$I = \int_{1/2}^{1} 12x\,dx = \frac{1}{4}[f(\frac{1}{2}) + f(1)]$

$= \frac{1}{4}[6 + 12] = \frac{18}{4}$

- For the complete interval $[0, 1]$

$I = \int_0^1 12x = \frac{18}{4} + \frac{16}{4} = \frac{24}{4} = 6$

b) using Simpson's rule over 2 uniform intervals

$m = 2 \quad n = 2$

- First interval $[0, \frac{1}{2}]$ $x_0 = 0 \quad x_1 = \frac{1}{4} \quad x_2 = \frac{1}{2}$

$I = \int_0^{1/2} 12x\,dx = \frac{1}{12}[f(0) + 4f(\frac{1}{4}) + f(\frac{1}{2})]$

$= \frac{1}{12}[0 + 12 + 6] = \frac{18}{12}$

- Secound interval $[\frac{1}{2}, 1]$ $x_0 = \frac{1}{2} \quad x_1 = \frac{3}{4} \quad x_2 = 1$

$I = \int_{1/2}^{1} 12x\,dx = \frac{1}{12}[f(\frac{1}{2}) + 4f(\frac{3}{4}) + f(1)]$

$= \frac{1}{12}[6 + 36 + 12] = \frac{54}{12}$

- For the compled interval $[0, 1]$ $I = \int_0^1 12x = \frac{18}{12} + \frac{54}{12} = 6$

(2) $\int_0^1 (5x^3+2x)\,dx$

a) using Trapezoidal rule Over 2 uniform interval

$m=2 \quad n=1$

- first interval $[0, \frac{1}{2}]$

$I = \int_0^{1/2} (5x^3+2x)\,dx = \frac{1}{4}[f(0) + f(\frac{1}{2})] = \frac{1}{4}[0 + \frac{13}{8}] = \frac{13}{32}$

- Second interval $[\frac{1}{2}, 1]$

$I = \int_{1/2}^{1} (5x^3+2) = \frac{1}{4}[f(\frac{1}{2}) + f(1)] = \frac{1}{4}[\frac{13}{8} + 7] = \frac{69}{32}$

- the complete interval

$I = \int_0^1 (5x^3+2) = \frac{13}{32} + \frac{69}{32} = \frac{82}{32} = \frac{41}{16} = 2.5625$

b) using Simpson's rule Over 2 uniform interval. $m=2$, $n=2$

- first interval $[0, \frac{1}{2}]$ $x_0 = 0$ $x_1 = \frac{1}{4}$ $x_2 = \frac{1}{2}$

$I = \int_0^{1/2} (5x^3+2x) = \frac{1}{4}[f(0) + 4f(\frac{1}{4}) + f(\frac{1}{2})]$

$= \frac{1}{12}[0 + 4\cdot\frac{37}{64} + \frac{13}{8}] = \frac{21}{64}$

- Secound interval $[\frac{1}{2}, 1]$ $x_0 = \frac{1}{2}$ $x_1 = \frac{3}{4}$ $x_2 = \frac{1}{2}$

$I = \int_{1/2}^{1} (5x^3+2x) = \frac{1}{4}[f(\frac{1}{2}) + 4f(\frac{3}{4}) + f(1)]$

$= \frac{1}{12}[\frac{13}{8} + 4\cdot\frac{231}{64} + 7] = \frac{123}{64}$

- the complete interval

$I = \int_0^1 (5x^3+2) = \frac{21}{64} + \frac{123}{64} = \frac{9}{4} = 2.25$

c) the exact solution

$\int_0^1 (5x^3+2x)\,dx = [\frac{5}{4}x^4 + x^2]_0^1 = \frac{9}{4}$

10. $\int_0^1\int_0^1 (9x^3+8x^2)(y^3+y)\,dx\,dy$

using Simpson' rule in each direction $n=2$

$x_0=0$ $\quad x_1=\frac{1}{2}$ $\quad x_2=1$

$y_0=0$ $\quad y_1=\frac{1}{2}$ $\quad x_2=1$

• ① putting $y=y_0=0$

$I_1=\int_0^1 f(x,0)\,dx=\frac{1}{6}[f(0,0)+4f(\frac{1}{2},0)+f(1,0)]=0$

• ② putting $y=y_1=0$

$I_2=\int_0^1 f(x,\frac{1}{2})\,dx=\frac{1}{6}[f(0,\frac{1}{2})+4f(\frac{1}{2},\frac{1}{2})+f(1,\frac{1}{2})]$

$=\frac{1}{6}[0+4.\frac{125}{64}+\frac{85}{8}]=\frac{265}{96}$

③ puttin $y=y_2=1$

$I_3=\int_0^1 f(x,1)\,dx=\frac{1}{6}[f(0,1)+4f(\frac{1}{2},1)+f(1,1)]$

$=\frac{1}{6}[0+4\frac{25}{4}+34]=\frac{56}{6}$

As For the y-direction.

$I=\frac{1}{6}[I_1+4I_2+I_3]=\frac{1}{6}[0+4.\frac{265}{96}+\frac{56}{6}]$

~~$I=\frac{476}{144}=3.326388$~~

$I=\frac{163}{46}=3.3958$

Error $=0.07837$