Exercise - 2

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1) Solution:

The motion of non-frictional pendulum is governed by ODE,

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

Where, θ = angular ~~velocity~~ displacement

L = 1m, length of pendulum

$g = 9.8\, m/s^2$, acceleration due to gravity

The initial boundary values,

The velocity and position at time $t = 1s$ are,

$$\frac{d\theta(1)}{dt} = 0 \text{ rad/s} \quad , \quad \theta(1) = 0.4 \text{ rad.}$$

Converting given 2^{nd} order ODE to system of 1^{st} order ODE's,

Let, $$\frac{d\theta}{dt} = z \quad \text{—— (1)}$$

$$\frac{dz}{dt} = \frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta \quad \text{—— (2)}$$

from eqn (1) and (2),

We have,

$$\frac{d\theta}{dt} = f(t, \theta, z) = z$$

$$\frac{dz}{dt} = g(t, \theta, z) = -\frac{g}{L}\theta$$

With, initial boundary conditions $\theta(1) = 0.4$ rad and $z(1) = 0$ rad/s

Second order Runge Kutta method is,
~~for backward~~ (for backward calculation h is negative)

$$z_{i+1} = z_i + \frac{h}{2}[l_1 + l_2]$$

where,

$$l_1 = g(t_i, \theta_i, z_i)$$
$$= -\frac{g}{L}\theta_i$$

$$l_2 = g(t_i + h, \theta_i + hk_1, z_i + hl_1)$$
$$= -\frac{g}{L}[\theta_i + k_1 h]$$

$$\theta_{i+1} = \theta_i + \frac{h}{2}[k_1 + k_2]$$

where

$$k_1 = f(t_i, \theta_i, z_i)$$
$$= z_i$$
$$k_2 = f(t_i + h, \theta_i + k_1 h, z_i + l_1 h)$$
$$= z_i + l_1 h$$

Using 2 steps

step size, $h = -0.5$, $t \in (0, 1)$

1st step.

for $i = 0$,

$z_0 = 0$, $\theta_0 = 0.4$

$$\theta_1 = \theta_0 + \frac{h}{2}[k_1 + k_2]$$

$$k_1 = z_0 = 0$$

$$k_2 = z_0 + l_1 h = 0 + l_1 \times (-0.5)$$
$$= -\frac{g\theta_i}{L}(-0.5)$$
$$= \frac{-9.8 \times 0.4 \times -0.5}{1}$$
$$= 1.96$$

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$\theta_1 = \theta_0 + \frac{h}{2}[k_1 + k_2]$

$= 0.4 - \frac{0.5}{2}[0 + 1.96]$

$= -0.09$ rad

$\theta(0.5) = -0.09$ rad

$z_1 = z_0 + \frac{h}{2}[l_1 + l_2]$

$l_1 = g(t_0, \theta_0, z_0)$

$= -9.8\theta_0$

$= -9.8 \times 0.4$

$= -3.92$

$l_2 = g(t_0 + h, \theta_0 + hk_1; z_0 + hl_1)$

$= -9.8(\theta_0 + hk_1)$

$= -9.8(0.4 - 0.5 \times 0)$

$= -3.92$

$z_1 = 0 - \frac{0.5}{2}[-3.92 - 3.92]$

$= 1.96$

2^{nd} step

for $i = 1$;

$(\theta_1 = -0.09, \cancel{\&}\ z_1 = 1.96)$

$z_2 = z_1 + \frac{h}{2}[l_1 + l_2]$

$\theta_2 = \theta_1 + \frac{h}{2}[k_1 + k_2]$

$k_1 = f(t_1, \theta_1, z_1)$

$= z_1$

$= -1.96$

$k_2 = f(t_1 + h, \theta_1 + k_1 h, z_1 + l_1 h)$

$= z_1 + l_1 h$

$= z_1 + h(-9.8 \times \theta_1)$

$= -1.96 - 0.5 \times (-9.8 \times -0.09)$

$= -1.519$

$$\theta_2 = \theta_{a1} + \frac{h}{2}[k_1 + k_2]$$

$$= -0.09 - \frac{0.5}{2}[1.96 + 1.519]$$

$$= -0.95975 \text{ rad}$$

$$\underline{\underline{\theta(0) = -0.95975 \text{ rad}}}$$

Using 4-steps,

Considering matrix approach,

The ODE is,

$$\theta'' + \frac{g}{L}\theta = 0$$

since $L = 1$,

$$\theta'' + g\theta = 0$$

Consider, $\theta_1 = \theta$

$\theta_2 = \theta'$

then,

$$\theta_1' = \theta' = \theta_2$$

$$\theta_2' = \theta'' = -g\theta_1$$

Now,

$$\begin{bmatrix} \theta_1' \\ \theta_2' \end{bmatrix} = \begin{bmatrix} \theta_2 \\ -g\theta_1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \theta_1' \\ \theta_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ g & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

This is of the form,

$$\vec{\theta}' = A\vec{\theta}$$

So here, $\vec{f}(\vec{\theta}, t) = A\vec{\theta}$

Initial boundary values, $\vec{\theta}(1) = \begin{bmatrix} 0.4 \\ 0 \end{bmatrix}$

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Runge Kutta method is,

$$\vec{\theta}_{i+1} = \vec{\theta}_i + \frac{h}{2}[\vec{K}_1 + \vec{K}_2]$$

$$K_1 = \vec{f}(t_i, \vec{\theta}_i) = \begin{bmatrix} \theta_2^{(i)} \\ -g\theta_1^{(i)} \end{bmatrix}$$

$$K_2 = \vec{f}(t_i+h, \vec{\theta}_i + h\vec{K}_1) = A\left\{\begin{bmatrix} \theta_1^{(i)} \\ \theta_2^{(i)} \end{bmatrix} + h\begin{bmatrix} \theta_2^{(i)} \\ -g\theta_1^{(i)} \end{bmatrix}\right\}$$

$$= A\begin{bmatrix} \theta_1^{(i)} + h\theta_2^{(i)} \\ \theta_2^{(i)} - hg\theta_1^{(i)} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -g & 0 \end{bmatrix}\begin{bmatrix} \theta_1^{(i)} + h\theta_2^{(i)} \\ \theta_2^{(i)} - hg\theta_1^{(i)} \end{bmatrix}$$

$$= \begin{bmatrix} \theta_2^{(i)} - hg\theta_1^{(i)} \\ -g(\theta_1^{(i)} + h\theta_2^{(i)}) \end{bmatrix}$$

$$\vec{\theta}_{(i+1)} = \vec{\theta}_{(i)} + \frac{h}{2}[\vec{K}_1 + \vec{K}_2]$$

$$\begin{bmatrix} \theta_1^{(i+1)} \\ \theta_2^{(i+1)} \end{bmatrix} = \begin{bmatrix} \theta_1^{(i)} \\ \theta_2^{(i)} \end{bmatrix} + \frac{h}{2}\left\{\begin{bmatrix} \theta_2^{(i)} \\ -g\theta_1^{(i)} \end{bmatrix} + \begin{bmatrix} \theta_2^{(i)} - hg\theta_1^{(i)} \\ -g(\theta_1^{(i)} + h\theta_2^{(i)}) \end{bmatrix}\right\}$$

$$= \begin{bmatrix} \theta_1^{(i)} + 0.5h(2\theta_2^{(i)} - hg\theta_1^{(i)}) \\ \theta_2^{(i)} - 0.5hg(2\theta_1^{(i)} + h\theta_2^{(i)}) \end{bmatrix}$$

For 4 steps,

we have $h = -\left(\frac{b-a}{4}\right) = \frac{-1}{4} = -0.25\text{s}$ ($h = -ve$, as it is backwards)

1^{st} step.

$i=0$, $(\theta_1^{(0)} = 0.4, \ \theta_2^{(0)} = 0)$

$$\vec{\theta}^{(1)} = \vec{\theta}^{(0)} + \frac{h}{2}[k_1 + k_2]$$

$$= \begin{bmatrix} \theta_1^{(0)} + 0.5h(2\theta_2^{(0)} - hg\theta_1^{(0)}) \\ \theta_2^{(0)} - 0.5gh(2\theta_1^{(0)} + h\theta_2^{(0)}) \end{bmatrix}$$

$$= \begin{bmatrix} 0.4 + 0.5 \times -0.25 \times (2\times 0 + 0.25 \times 9.8 \times 0.4) \\ 0 + 0.5 \times 9.8 \times 0.25\,(2\times 0.4 + 0.25 \times 0) \end{bmatrix}$$

$$\begin{bmatrix} \theta_1^{(1)} \\ \theta_2^{(1)} \end{bmatrix} = \begin{bmatrix} 0.2775 \\ 0.98 \end{bmatrix}$$

2nd step,

For $i=1$, $(\theta_1^{(1)} = 0.2775,\ \theta_2^{(1)} = 0.98)$

$$\begin{bmatrix} \theta_1^{(2)} \\ \theta_2^{(2)} \end{bmatrix} = \begin{bmatrix} 0.2775 - 0.5\times 0.25(2\times 0.98 + 0.25 \times 0.98 \times 0.2775) \\ 0.98 + 0.5 \times 9.8 \times 0.25(2\times 0.2775 - 0.25 \times 0.98) \end{bmatrix}$$

$$= \begin{bmatrix} -0.052484 \\ 1.35975 \end{bmatrix}$$

3rd step

For $i=2$,

$(\theta_1^{(2)} = -0.052484,\ \theta_2^{(2)} = 1.35975)$

$$\begin{bmatrix} \theta_1^{(3)} \\ \theta_2^{(3)} \end{bmatrix} = \begin{bmatrix} -0.052484 + 0.5 \times -0.25(2\times 1.35975 + 0.25 \times 9.8 \times -0.052484) \\ 1.35975 - 0.5 \times 9.8 \times 0.25(2\times -0.052484 - 0.25 \times 1.3975) \end{bmatrix}$$

$$\begin{bmatrix} \theta_1^{(3)} \\ \theta_2^{(3)} \end{bmatrix} = \begin{bmatrix} -0.37635 \\ 0.814741 \end{bmatrix}$$

4th step

for $i=3$,

$(\theta_1^{(3)} = -0.37635 \quad \theta_2^{(3)} = 0.814741)$

$$\vec{\theta}^{(4)} = \vec{\theta}^{(3)} + \frac{h}{2}[k_1 + k_2]$$

$$\Rightarrow \begin{bmatrix} \theta_1^{(4)} \\ \theta_2^{(4)} \end{bmatrix} = \begin{bmatrix} -0.37635 - 0.5\times 0.25(2\times 0.814774 + 0.25 \times 9.8\times -0.37635) \\ 0.814774 + 0.5\times 9.8\times 0.25(2\times -0.37635 - 0.25\times 0.814774) \end{bmatrix}$$

$$= \cancel{\begin{bmatrix} -0.46464 \\ -0.35723 \end{bmatrix}} = \begin{bmatrix} -0.464778 \\ -0.35683 \end{bmatrix}$$

$$\theta_1^{(4)} = \theta(0) = -0.464778$$

At $t=0$,

Using 2 time steps $\theta(0) = -0.95975$ rad

Using 4 time steps $\theta(0) = -0.464778$ rad

b) To find relative error.

$$\text{Relative error} = \left| \frac{\text{Measured value} - \text{Actual value}}{\text{Actual value}} \right|$$

	By 2 step method	Actual value	Relative error
$\theta(0.5)$	-0.09	0.002219	39.56
$\theta(0)$	-0.95975	-0.39998	1.399

To find actual values,

$$\theta'' + g\theta = 0$$

comparing with, $r^2 + gr = 0$

determinant, $b^2 - 4ac = 0^2 - 4\times 1\times 9.8 < 0$.

Solution is of the form,

$$\theta = C_1 e^{\alpha t} \cos(\beta t) + C_2 e^{\alpha t} \sin(\beta t)$$

$$\alpha = \frac{-b}{2a} = 0$$

$$\beta = \frac{\sqrt{4ac - b^2}}{2a} = \frac{\sqrt{4g}}{2} = \sqrt{g}$$

$$\theta_1 = C_1\, e^0 \cos(\sqrt{g}\,t) + C_2\, e^0 \sin(\sqrt{g}\,t)$$

$$\theta_1 = C_1 \cos(\sqrt{g}\,t) + C_2 \sin(\sqrt{g}\,t)$$
$$\frac{d\theta}{dt} = -C_1\sqrt{g}\sin(\sqrt{g}\,t) + C_2\sqrt{g}\cos(\sqrt{g}\,t)$$

Applying B.C,

$$\theta(1) = C_1\cos\sqrt{g} + C_2\sin\sqrt{g} = 0.4 \quad -①^*$$

$$\theta'(1) = (-C_1\sin\sqrt{g} + C_2\cos\sqrt{g})\sqrt{g} = 0 \quad -②^*$$

$$\Rightarrow C_1\sin\sqrt{g} = C_2\cos\sqrt{g}$$

$$C_2 = \frac{C_1\sin\sqrt{g}}{\cos\sqrt{g}}$$

substituting in ①*,

$$C_1\cos\sqrt{g} + C_1\frac{\sin\sqrt{g}}{\cos\sqrt{g}}\sin\sqrt{g} = 0.4$$

$$\frac{C_1(\cos^2\sqrt{g} + \sin^2\sqrt{g})}{\cos\sqrt{g}} = 0.4$$

$$C_1 = 0.4\cos\sqrt{g}$$

$$\Rightarrow C_2 = 0.4\cos\sqrt{g} \times \frac{\sin\sqrt{g}}{\cos\sqrt{g}}$$

$$= 0.4\sin\sqrt{g}$$

$$\theta = 0.4\left[\cos\sqrt{g}\cos(\sqrt{g}\,t) + \sin\sqrt{g}\sin(\sqrt{g}\,t)\right]$$

$$= 0.4\cos(\sqrt{g}(t-1)]$$

So, now

$$\theta(t=0.5) = 0.4\cos(\sqrt{g}\,(0.5-1))$$

$$= \underline{0.002219} \text{ rad}$$

$$\theta(t=0.5) = 0.4\cos(\sqrt{g}\,(0-1))$$

$$= \underline{0.39998} \text{ rad}$$

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The approximate relative error of solution between 2 step approximation and analytical solution is, 1.399 or 139.9%.

The approximate relative error of solution between 2 step and 4 step approximations is,

$$RE = \left| \frac{[\theta(0)]_{2\text{-step}} - [\theta(0)]_{4\text{-step}}}{[\theta(0)]_{4\text{step}}} \right|$$

$$= \left| \frac{-0.95975 + 0.464778}{-0.464778} \right|$$

$= 1.065$
or
$= \underline{106.5\%}$

c) To propose a time step h to obtain error of three magnitudes less, we will use the error formula,

$$E_h = Ch^{P+1}$$

where, P is order
h is step size.

Given, $E_h^* = 10^{-3} E_h$.

$$\frac{E_h^*}{E_h} = \frac{C(h^*)^{P+1}}{Ch^{P+1}} = \frac{(h^*)^3}{(h)^3}$$

$$h^* = \left(\frac{E_h^*}{E_h}\right)^{1/3} h$$

$$h^* = (10^{-3})^{1/3} \times 0.5 = \underline{0.05}$$

$h^* = 0.05$

The time steps

$$= \frac{b-a}{h^*}$$

$$= \frac{1-0}{0.05}$$

$= \underline{20 \text{ steps}}$

4

2) Given ODE is,

$$\frac{dy}{dx} = y - x^2 + 1 \qquad x \in (0,1)$$

$y(0) = 1$

So, $f(x,y) = \frac{dy}{dx} = y - x^2 + 1$

a) Using Euler method, with $h = 0.25$

$$y_{i+1} = y_i + hf(x_i, y_i)$$

1st step.

$i=0$, $(y_0 = 1,\ x_0 = 0.00)$

$y_1 = y_0 + hf(x_0, y_0)$

$= 1 + 0.25 \times (y_0 - x_0^2 + 1)$

$= 1 + 0.25 \times (1 - 0 + 1)$

$= 1 + 0.25 \times 2$

$y(0.25) = 1.5$

2nd step

$i=1$, $(y_1 = 1.5,\ x_1 = 0.25)$

$y_2 = y_1 + hf(x_1, y_1)$

$= 1.5 + 0.25\, f(0.25, 1.5)$

$= 1.5 + 0.25\,(1.5 - 0.25^2 + 1)$

$y(0.5) = 2.1094$

3rd step

$i=2$ $(y_2 = 2.1094,\ x_2 = 0.5)$

$y_3 = y_2 + hf(x_2, y_2)$

$= 2.1094 + 0.25\,(2.1094 - 0.5^2 + 1)$

$y(0.75) = 2.8242$

4^{st} step

$i=3$ $(Y_3 = 2.8242,\ X_3 = 0.75)$

$$Y_4 = Y_3 + h\left(f(x_3, Y_3)\right)$$

$$= 2.8242 + 0.25\left(2.8242 - 0.75^2 + 1\right)$$

$$Y(1) = \underline{3.6396}$$

b) Heun method with same computational cost or iterations (steps)

$h = 0.25$

$$Y_{i+1} = Y_i + \frac{h}{2}[k_1 + k_2]$$

$$k_1 = f(x_i, Y_i)$$

$$k_2 = f(x_i + h, Y_i + hk_1)$$

$$\frac{dy}{dx} = y - x^2 + 1 \qquad x \in (0,1)$$

$$Y_0 = 1$$

So, $f(x, Y) = \frac{dy}{dx} = y - x^2 + 1$

1st step

$i=0$, $(Y_0 = 1,\ x_0 = 0)$

$$k_1 = f(0, 1)$$

$$= 1 - 0 + 1$$

$$= 2$$

$$k_2 = f(x_0 + h, Y_0 + hk_1)$$

$$= f(0 + 0.25,\ 1 + 0.25 \times 2)$$

$$= f(0.25, 1.5)$$

$$= 1.5 - 0.25^2 + 1$$

$$= \underline{2.4375}$$

$$Y_1 = Y_0 + \frac{h}{2}[k_1 + k_2]$$

$$= 1 + \frac{0.25}{2}[2 + 2.4375]$$

$$Y(0.25) = 1.\underline{5547}$$

2^{nd} step

$i=1, \quad Y_1 = 1.5547, \quad x_1 = 0.25$

$$k_1 = f(x_1, Y_1) = f(0.25, 1.5547)$$
$$= 1.5547 - 0.25^2 + 1$$
$$= 2.4922$$

$$k_2 = f(x_1 + h, Y_1 + hk_1)$$
$$= f(0.25 + 0.25, 1.5547 + 0.25 \times 2.4922)$$
$$= f(0.5, 2.1777)$$
$$= 2.1777 - 0.5^2 + 1$$
$$= 2.9277$$

$$Y_2 = Y_1 + \frac{h}{2}[k_1 + k_2]$$
$$= \overset{1.5547}{\cancel{2.9277}} + \frac{0.25}{2}[2.4922 + 2.9277]$$
$$Y(0.5) = 2.2322$$

3^{rd} step

$i=2, \quad Y_2 = 2.2322, \quad x_2 = 0.5$

$$k_1 = f(x_2, Y_2)$$
$$= f(0.5, 2.2322)$$
$$= 2.2322 - 0.5^2 + 1$$
$$= 2.9822$$

$$k_2 = f(x_2 + h, Y_2 + hk_1)$$
$$= f(0.5 + 0.25, 2.2322 + 0.25 \times 2.9822)$$
$$= f(0.75, 2.9777)$$
$$= 2.9777 - 0.75^2 + 1$$
$$= 3.4152$$

$$Y_3 = Y_2 + \frac{h}{2}[k_1 + k_2]$$
$$= 2.2322 + \frac{0.25}{2}[2.9822 + 3.4152]$$
$$Y(0.75) = 3.0319$$

$\underline{4^{th} \text{ step}}$

$i=3,\ x_3=0.75,\ Y_3=3.0319$

$$Y_4 = Y_3 + \frac{h}{2}[k_1+k_2]$$

$$k_1 = f(0.75, 3.0319)$$
$$= [3.0319 - 0.75^2 + 1]$$
$$= 3.4694$$

$$k_2 = f(x_3+h,\ Y_3+hk_1)$$
$$= f(0.75+0.25,\ 3.0319+0.25\times 3.4694)$$
$$= f(1, 3.8992)$$
$$= 3.8992 - 1^2 + 1$$
$$= \underline{\underline{3.8992}}$$

$$Y_4 = 3.0319 + \frac{0.25}{2}[3.4694 + 3.8992]$$
$$Y(1) = \underline{\underline{3.9530}}$$

Results,

y	From Euler method	From (Heun) Runge Kutta method
$Y(0.25)$	1.5	1.5547
$Y(0.5)$	2.1094	2.2322
$Y(0.75)$	2.8242	3.0319
$Y(1)$	3.6396	3.9530

3) <u>Solution</u>

$$\frac{dy}{dx} = f(x,y)$$
$$x \in (0,1)$$
$$y(0)=1$$

Approximating, taylor series to $O(h^2)$, we have truncated taylor series

$y_{i+1} = y(x_i + h)$

$= y(x_i) + h y'(x_i) + \frac{h^2}{2!} y''(x_i)$

$y_{i+1} = y_i + h y_i' + \frac{h^2}{2} y_i''$

$h y_i' = y_{i+1} - y_i - \frac{h^2}{2} y_i''$

$\Rightarrow \quad f(x_i, y_i) = \frac{y_{i+1} - y_i}{h} - \frac{h}{2} y''(x_i)$

$f(x_i, y_i) = \frac{y_{i+1} - y_i}{h} - \tau_i(h)$

So, the truncation error is $\tau_i(h) \approx -\frac{h}{2} y''(x_i)$

Checking consistency of method,

$\max\limits_{\substack{\text{when } h \to 0 \\ 0 \le i \le m}} (\tau_i(h)) = \lim\limits_{h \to 0} -\frac{h}{2} y''(x_i) \approx \underline{\underline{0}}$

$\max(\tau_i(h)) \to 0$, when $h \to 0$ therefore, the method is consistent.

b) Backward Euler method.

$f(x, y) = \frac{dy}{dx}$, is a non linear function of x and y.

From Taylor series,

$y(x_i) = y(x_{i+1} - h)$

$= y(x_{i+1} + (-h))$

$\Rightarrow y(x_i) = y(x_{i+1}) + \frac{y'(x_{i+1})}{1!} \cdot (-h) + \frac{h^2}{2!} y''(x_{i+1})$

$$y_i = y_{i+1} - h f(x_{i+1}, y_{i+1}) + \frac{h^2}{2} y''(x_{i+1})$$

here

$$\Rightarrow f(x_{i+1}, y_{i+1}) = \frac{dy}{dx}(x_{i+1}, y_{i+1}) = \frac{y_{i+1} - y_i}{h} + \frac{h}{2} y''(x_{i+1})$$

Truncation error, $\tau_i(h) = \frac{h}{2} y''(x_{i+1})$

The Euler backward method formula is

$$Y_{i+1} = Y_i + h f(x_{i+1}, Y_{i+1})$$

It's an implicit method.

c) Given,

$$\frac{dy}{dx} = f(x, y) = -\lambda y$$

Euler Forward method for $f(x, y) = -\lambda y$ is,

$$Y_{i+1} = Y_i - h\lambda Y_i$$

$$Y_{i+1} = (1 - h\lambda) Y_i$$

$$Y_{i+1} = G Y_i$$

Aplification factor, $G = 1 - h\lambda$

The scheme is stable if $|G| < 1$,

$$|1 - h\lambda| < 1$$

Stability limits of λ are,

$$-1 < 1 - h\lambda < 1$$

$$-2 < -h\lambda < 0$$

$\boxed{2 > h\lambda > 0}$ $\longrightarrow$ It's conditionally stable.

For Backward Euler method,

$$f(x,y) = -\lambda y$$

$$y_{i+1} = y_i - h\lambda\, y_{i+1}$$

$$y_{i+1}(1+h\lambda) = y_i$$

$$y_{i+1} = \frac{1}{1+h\lambda}\, y_i$$

Amplification factor $G = \frac{1}{1+h\lambda}$

Stability condition is,

$$|G| < 1$$

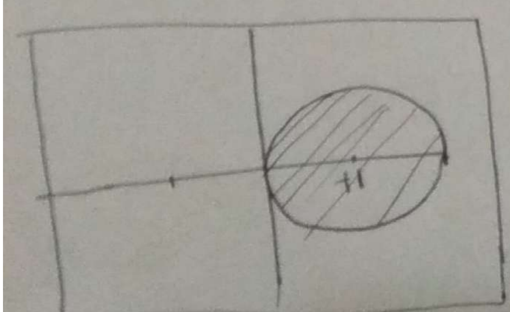
$$\left|\frac{1}{1+h\lambda}\right| < 1$$

$$-1 < \frac{1}{1+h\lambda} < 1$$

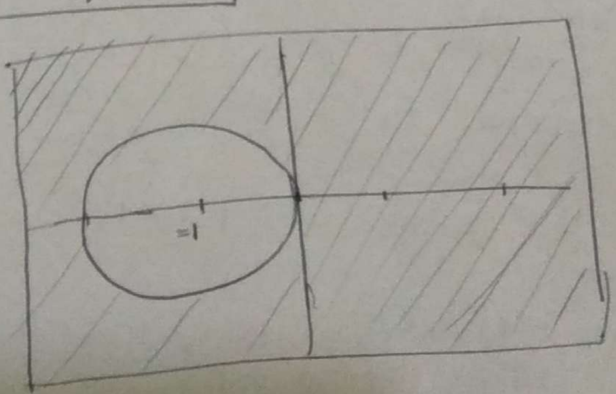
$$-1 > 1+h\lambda > 1$$

$$-2 > h\lambda > 0$$

$\boxed{h\lambda > 0 \quad \text{or} \quad h\lambda < -2}$ Unconditionally stable for $(\lambda > 0)$



$|1 \mp h\lambda| < 1$



$|1+h\lambda| > 1$

d)

$$\frac{dy}{dx} = -2.5\, y^{3.5}$$

$$y(0) = 1$$

Using ~~Forward~~ Backward Euler method with $h = 1/10$

$$Y_{i+1} = Y_i + hf(X_{i+1}, Y_{i+1})$$

1st step

$i = 0, \quad h = 1/10, \quad y_0 = 1, \quad x_0 = 0$

$$y_1 = y_0 + hf(x_1, y_1)$$

$$y_1 = 1 + \tfrac{1}{10} f(\tfrac{1}{10}, y_1)$$

$$y_1 = 1 + \frac{1}{10}\left[-25\, y_1^{3.5}\right]$$

$$y_1 = 1 - 2.5\, y_1^{3.5}$$

$$2.5\, y_1^{3.5} + y_1 - 1 = 0 = g(y_1)$$

$$\Rightarrow g'(y) = 8.75\, y_1^{2.5} + 1$$

Solving for y_1 using Newton's method, with initial approximation,

$$(y_1)_0 = 0.5$$

1st iteration,

$$(y_1)_1 = (y_1)_0 - \frac{g(y_1)_0}{g'(y_1)_0}$$

$$= 0.5 - \frac{2.5 \times (0.5)^{3.5} + 0.5 - 1}{8.75 \times 0.5^{2.5} + 1}$$

$$= 0.6096$$

2nd iteration,

$$(y_1)_2 = (y_1)_1 - \frac{g(y_1)_1}{g'(y_1)_1}$$

$$= 0.60956 - \frac{2.5\times(0.60956)^{3.5} + 0.60956 - 1}{8.75\times(0.60956)^{2.5} + 1}$$

$(y_1)_2 = y(1/10) = 0.5950$

$\therefore\ y(1/10) = 0.5950$

2nd step.

$i=2,\ y_1 = 0.5950,\ x_1 = 1/10$

$$y_2 = y_1 + hf(x_2, y_2)$$

$$y_2 = 0.5950 + \frac{1}{10}\left[-25\, y_2^{3.5}\right]$$

$$y_2 = 0.5950 - 2.5\, y_2^{3.5}$$

$$2.5\, y_2^{3.5} + y_2 - 0.5950 = 0 = g(y_2)$$

$$g'(y_2) = 8.75\, y_2^{2.5} + 1$$

Using Newton method to solve for y_2, with initial approximation 0.4

$\therefore\ (y_2)_0 = 0.4$

$$(y_2)_1 = (y_2)_0 - \frac{g(y_2)_0}{g'(y_2)_0}$$

$$= 0.4 - \frac{2.5(0.4)^{3.5} + 0.4 - 0.5950}{8.75\times 0.4^{2.5} + 1}$$

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$(y_2)_1 = 0.4498$

2^{nd} iteration.

$$(y_2)_2 = (y_2)_1 - \frac{g(y_2)_1}{g'(y_2)_1}$$

$$= 0.4498 - \frac{2.5 \times 0.4498^{(3.5)} + 0.4498 - 0.5950}{8.75 \times 0.4498^{2.5} + 1}$$

$(y_2)_2 = 0.4465$

$\therefore y(\frac{2}{10}) = 0.4465$

e> Solving.

$$\frac{dy}{dx} = -25y^{3.5}$$

$$y(0) = 1$$

Using Forward Euler method.

$$Y_{i+1} = Y_i + hf(X_i, Y_i)$$

1^{st} step,

for $i=0$, $h = \frac{1}{10}$

$Y_0 = 1$, $X_0 = 0$

$$Y_1 = Y_0 + hf(X_0, Y_0)$$

$$= 1 + \frac{1}{10}(-25 \times 1^{3.5})$$

$$= 1 - \frac{25}{10}$$

$$= 1 - 2.5$$

$Y_0 = -1.5$

$\therefore Y(1/10) = -1.5$

2^{nd} iteration

$$Y_2 = Y_1 + h f(x_1, Y_1)$$
$$= -1.5 + \frac{1}{10}\left(-25 \times (-1.5)^{3.5}\right)$$
$$= -1.5 - 2.5(0 - 4.1335i)$$
$$= -1.5 + 10.338i \quad \text{(Not real)}$$

Solution not stable for this method

f) For Forward Euler method,
Using matlab the given problem is solved for step sizes $h = 1/10$, $h = 1/15$, $h = \frac{1}{30}$, $h = 1/45$, $h = 1/90$.

For $h = 1/10$ and $h = 1/15$, the solution is imaginary and the solution is not stable. It goes outside of the method is not the domain.

For $h = 1/30, 1/45, \frac{1}{90}$ the solution is real and method is stable (The solution is within domain)

Hence, maximum stable interval size is $1/30$ (among given)

Using matlab the stability condition of step size h is found as

$$h \leq \frac{1}{25} \quad \boxed{\cancel{0.04}\ h}$$

Conceptually,

$$Y_{i+1} = Y_i + h f(x_i, Y_i)$$
$$Y_{i+1} = Y_i - 25h\,Y_i^{3.5}$$
$$Y_{i+1} = \left|1 - 25h\,Y_i^{2.5}\right| Y_i$$

for stability,

Amplification factor, $|G| < 1$

$$\left|1 - 25h\,Y_i^{2.5}\right| < 1$$
$$(1 - 25h\,Y_i^{2.5}) < 1$$
$$-1 < (1 - 25h\,Y_i^{2.5})$$

$$-2 < -25 h Y_i^{2.5} < 0$$

$\boxed{\cancel{\text{If } Y_i \text{ is}}}$

Y_i is positive with domain $(0,1)$

Hence, $\dfrac{-2}{25 Y_i^{2.5}} < -h < 0$

$$\frac{2}{25 Y_i^{2.5}} > h > 0$$

Considering maximum $Y_i = 1$,

$$\Rightarrow \quad \frac{2}{25} > h > 0$$

$$\Rightarrow \quad h < \frac{2}{25}$$

Since, the $\frac{dy}{dx}$ is not linear interms of y, the stability condition is not matching with the actual maximum ~~stability~~ stable step size of h.

i.e $\left(h \leq \frac{1}{25}\right)$