Numerical Methods for Partial Differential Equations - ODEs

1. The motion of a non-frictional pendulum is governed by the ordinary differential equation (ODE)

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

where θ is the angular displacement, $L = 1\,m$ is the pendulum length and the gravity acceleration is $g = 9.8\,m/s^2$.

The position and velocity at time $t = 1\,s.$ are known:

$$\theta(1) = 0.4\ rad, \qquad \frac{d\theta}{dt}(1) = 0\ rad/s$$

a) Solve the initial boundary value problem in the interval $(0,1)$ using a second-order Runge-Kutta method to determine the initial position at $t = 0\,s.$, with 2 and 4 time steps.

b) Using the approximations obtained in a), compute an approximation of the relative error in the solution computed with 2 steps.

c) Propose a time step h to obtain an approximation with a relative error three orders of magnitude smaller.

Solution - (a)

Rewrite the second-order ordinary differential equation as a system of two first-order ODEs

$$\frac{d\theta}{dt} = S \qquad \theta(1) = 0.4$$

$$\frac{dS}{dt} = -\frac{g}{L}\theta \qquad S(1) = 0$$

The second-order Runge-Kutta method is given by

$$y_{i+1} = y_i + \frac{h}{2}(k_1 + k_2) \quad \text{with } k_1 = f(t_i, y_i) \text{ and } k_2 = f(t_i + h, y_i + hk_1)$$

solution - (a) cont.

Solving the system of first-order ODEs by the second-order Runge-Kutta method

$$\theta_{i+1} = \theta_i + \frac{h}{2}\left[s_i + (s_i + hs_i)\right] = \theta_i + \frac{1}{2}h(h+2)s_i \ ,$$

$$s_{i+1} = s_i + \frac{h}{2}\left[-\frac{g}{L}\theta_i - \frac{g}{L}\left(\theta_i - \frac{g}{L}h\theta_i\right)\right] = s_i - \frac{1}{2}h\frac{g}{L}\left(2 - \frac{g}{L}h\right)\theta_i \qquad \text{where } g = 9.8 \ , L = 1$$

Solve the system with 2 time steps with $h = 0.5$

Assume $\theta(0)^{(1)} = 0 \quad , \quad s(0)^{(1)} = -0.1$

$$\theta_1 = \theta_0 + 0.25(2.5)s_0 = -0.0625$$

$$s_1 = s_0 + 7.105\,\theta_0 = -0.1$$

$$\theta_2 = \theta_1 + 0.25(2.5)s_1 = -0.125$$

$$s_2 = s_1 + 7.105\,\theta_1 = -0.54406$$

Assume $\theta(0)^{(2)} = 0.4 \quad , \quad s(0)^{(2)} = 0$

$$\theta_1 = \theta_0 + 0.25(2.5)s_0 = 0.4$$

$$s_1 = s_0 + 7.105\,\theta_0 = 2.842$$

$$\theta_2 = \theta_1 + 0.625\,s_1 = 2.17625$$

$$s_2 = s_1 + 7.105\,\theta_1 = 5.684$$

Use secant method to find next initial conditions ;

$$\text{slope}_1 = \frac{\theta(1)^{(2)} - \theta(1)^{(1)}}{s(0)^{(2)} - s(0)^{(1)}} = \frac{2.17625 + 0.125}{0 + 0.1} = 23.0125$$

$$s(0)^{(3)} = s(0)^{(2)} + \frac{\theta(1) - \theta(1)^{(2)}}{\text{slope}} = 0 + \frac{0.4 - 2.17625}{23.0125} = -0.07719$$

$$\text{slope}_2 = \frac{\theta(1)^{(2)} - \theta(1)^{(1)}}{\theta(0)^{(2)} - \theta(0)^{(1)}} = \frac{2.17625 + 0.125}{0.4 - 0} = 5.753125$$

$$\theta(0)^{3} = \theta(0)^{(2)} + \frac{\theta(1) - \theta(1)^{(2)}}{\text{slope}} = 0.4 + \frac{0.4 - 2.17625}{5.753125} = 0.09125$$

solution - (a) cont.

Assume $\theta(0)^{(3)} = 0.09125$, $S(0)^{(3)} = -0.07719$

$\theta_1 = \theta_0 + 0.625 S_0 = 0.04301$

$S_1 = S_0 + 7.105\theta_0 = 0.57114$

$\theta_2 = \theta_1 + 0.625 S_1 = 0.39997$

$S_2 = S_1 + 7.105\theta_1 = 0.87673$

Solve the system with 4 time steps with $h = 0.25$ $\left[\begin{array}{l} \theta_{i+1} = \theta_i + 0.28125 S_0 \\ S_{i+1} = S_i + 0.55125\theta_i \end{array}\right]$

Assume $\theta(0)^{(1)} = 0$, $S(0)^{(1)} = -0.1$

$\theta_1 = -0.028125$, $S_1 = -0.1$

$\theta_2 = -0.05625$, $S_2 = -0.11550$

$\theta_3 = -0.08874$, $S_3 = -0.14651$

$\theta_4 = -0.12994$, $S_4 = -0.19543$

Assume $\theta(0)^{(2)} = 0.4$, $S(0)^{(2)} = 0$

$\theta_1 = 0.4$, $S_1 = 0.2205$.

$\theta_2 = 0.46202$, $S_2 = 0.441$

$\theta_3 = 0.58605$, $S_3 = 0.69569$

$\theta_4 = 0.78171$, $S_4 = 1.01874$

Use secant method to find next initial conditions

$$\text{slope}_1 = \frac{\theta(1)^{(2)} - \theta(1)^{(1)}}{S(0)^{(2)} - S(0)^{(1)}} = \frac{0.78171 + 0.12994}{0 + 0.1} = 9.1165$$

$$S(0)^{(3)} = S(0)^{(2)} + \frac{\theta(1) - \theta(1)^{(2)}}{\text{slope}} = 0 + \frac{0.4 - 0.78171}{9.1165} = -0.04187$$

$$\text{slope}_2 = \frac{\theta(1)^{(2)} - \theta(1)^{(1)}}{\theta(0)^{(2)} - \theta(0)^{(1)}} = \frac{0.78171 + 0.12994}{0.4 - 0} = 2.279125$$

$$\theta(0)^{(3)} = \theta(0)^{(2)} + \frac{\theta(1) - \theta(1)^{(2)}}{\text{slope}} = 0.4 + \frac{0.4 - 0.78171}{2.279125} = 0.23252$$

Solution - (a) cont.

Assume $\theta(0)^{(3)} = 0.23252$, $S(0)^{(3)} = -0.04187$

$\theta_1 = 0.22074$, $S_1 = 0.08631$

$\theta_2 = 0.24502$, $S_2 = 0.20799$

$\theta_3 = 0.30352$, $S_3 = 0.34306$

$\theta_4 = 0.40000$, $S_4 = 0.51037$

The initial position at $t=0$ from using 2 time steps is 0.09125 rad

The initial position at $t=0$ from using 4 time steps is 0.23252 rad.

Solution - (b) Use the values from using 4 time steps as reference.

At $t=0$, relative error $= \dfrac{0.09125 - 0.23252}{0.23252} = -0.60756$

At $t=0.5$, relative error $= \dfrac{0.04301 - 0.30352}{0.30352} = -0.85830$

Solution - (c)

We are using the second-order method. That means the local truncation error is in the order of $O(h^3)$.

Local error: $E_h = Ch^{p+1}$ where $p=2$ in this case.

Error under a prescribed tolerance: $E_{h^*} = C(h^*)^{p+1} \leq tol$

Looking for a relative error three orders of magnitude smaller $E_{h^*} = \frac{1}{2}E_h$

$$\frac{E_h^*}{E_h} = \frac{\cancel{c}(h^*)^3}{\cancel{c}(h)^3} = \frac{1}{2}$$

$$h^* = \frac{1}{\sqrt[3]{2}}h$$

2. Consider the initial value problem

$$\frac{dy}{dx} = y - x^2 + 1 \qquad x \in (0,1)$$

$$y(0) = 1$$

a) Solve the initial value problem using the Euler method with step $h = 0.25$.

b) Compute the solution using the Heun method with a step h such that the computational cost is equivalent to the computational cost in a)

Note that the analytical solution of the initial value problem is a second degree polynomial

c) Compute the pure interpolation polynomial that fits the results in b)

d) Which approximation criterion do you recommend to fit the results obtained in a)? Compute the polynomial approximation with the proposed criterion and compare the results with the polynomial obtained in c)

solution - (a)

The derivative function $f(x_i, y_i) = y_i - x_i^2 + 1$

The Euler method is given by

$$y_{i+1} = y_i + h f(x_i, y_i) \qquad \text{where } h = 0.25$$

For the first step $i=0$

$$y_1 = y_0 + 0.25 f(x_0, y_0) = y_0 + 0.25(y_0 - x_0^2 + 1)$$

$$= 1 + 0.25(1 - 0^2 + 1)$$

$$= 1.5$$

For the second step $i = 1$

$$y_2 = y_1 + 0.25 f(x_1, y_1) = y_1 + 0.25(y_1 - x_1^2 + 1)$$

$$= 1.5 + 0.25(1.5 - 0.25^2 + 1)$$

$$= 2.109375$$

For the third step $i = 2$

$$y_3 = y_2 + 0.25 f(x_2, y_2) = y_2 + 0.25(y_2 - x_2^2 + 1)$$

$$= 2.109375 + 0.25(2.109375 - 0.5^2 + 1)$$

$$= 2.82419375$$

solution - (a) cont.

For the fourth step $i = 3$

$$y_4 = y_3 + 0.25 f(x_3, y_3) = y_3 + 0.25(y_3 - x_3^2 + 1)$$

$$= 2.82419375 + 0.25(2.82419375 - 0.75^2 + 1)$$

$$= 3.6396171875$$

solution - (b)

The computational cost is..

$$\text{cost} = n_f N = \frac{n_f t_f}{h}$$

where n_f = number of evaluations of function f per step

N = number of time steps

t_f = final time

h = step size

Therefore, to compare the Euler and Heun methods at the same computational cost, a time step twice as large must be used for Heun as Euler method requires just one evaluation of f per step while the Heun method requires two.

The Heun method is given by

$$y_{i+1} = y_i + \frac{h}{2}(k_1 + k_2) \quad \text{with } k_1 = f(x_i, y_i) \text{ and } k_2 = f(x_i + h, y_i + hk_1)$$

Applying the Heun method to this problem with 2 time steps where $h = 0.5$

For the first step, we have

$$y_1 = y_0 + 0.25(k_1 + k_2)$$

where $k_1 = f(x_0, y_0)$

$$= y_0 - x_0^2 + 1$$

$$= 1 - 0^2 + 1 = 2$$

Solution -(b) cont.

$$k_2 = f(x_0+0.5,\ y_0+0.5k_1)$$

$$= (y_0+0.5k_1) - (x_0+0.5)^2 + 1$$

$$= (1+0.5(2)) - 0.5^2 + 1 = 2.75$$

Substituting k_1, k_2 and y_0

$$y_1 = y_0 + 0.25(2+2.75)$$

$$= 2.1875$$

For the second step, we have

$$y_2 = y_1 + 0.25(k_1+k_2)$$

where $k_1 = f(x_1, y_1)$

$$= y_1 - x_1^2 + 1$$

$$= 2.1875 - 0.5^2 + 1 = 2.9375$$

$$k_2 = f(x_1+0.5,\ y_1+0.5k_1)$$

$$= (y_1+0.5k_1) - (x_1+0.5)^2 + 1$$

$$= 3.65625$$

Substituting k_1, k_2 and y_1

$$y_2 = y_1 + 0.25(2.9375 + 3.65625)$$

$$= 3.8359375$$

Solution - (c)

From Heun method, we get

x	y
0	1
0.5	2.1875
1	3.8359375

Suppose that the interpolation polynomial is in the form

$$p(x) = a_2 x^2 + a_1 x + a_0$$

At each data point x_i

$$p(x_i) = y_i$$

Then we can obtain a system of linear equations in the matrix form

$$\begin{bmatrix} x_0^2 & x_0 & 1 \\ x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \end{bmatrix} \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}$$

Using x and y data points from Heun method yields

$$\begin{bmatrix} 0 & 0 & 1 \\ 0.25 & 0.5 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2.1875 \\ 3.8359375 \end{bmatrix}$$

Solving the matrix yields $a_0 = 1$, $a_1 = 1.914$, $a_2 = 0.9219$

Therefore, the polynomial fitting the results in b) is

$$y = 0.9219x^2 + 1.9141x + 1$$

3. The ordinary differential equation

$$\frac{dy}{dx} = f(x,y)$$

is defined over the domain $(0,1)$, and is to be solved numerically subject to the initial condition $y(0) = 1$, where $y(x)$ is the exact solution. The forward Euler method for integrating the above differential equation is written as

$$Y_{i+1}' = Y_i + hf(x_i, Y_i)$$

where Y_i denotes the discrete solution at node i, with position x_i, for a uniform grid of nodes of constant grid interval size h and $x_{i+1} = x_i + h$.

a) Using a Taylor series expansion, deduce the leading truncation error of the scheme. Is the method consistent? Explain your answer.

b) State the backward Euler method for integrating the above differential equation where $f(x,y)$ is a general non-linear function of x and y.

c) Deduce the stability limits of the respective forward Euler method and backward Euler method for the model equation $dy/dx = -\lambda y$ where λ is a positive real constant.

d) Use the backward Euler method to compute the numerical solution of the ordinary differential equation

$$\frac{dy}{dx} = -25y^{3.5}$$

with initial condition $y(0) = 1$, by hand for two steps with grid interval size $h = 1/10$. (Use 2 Newton iterations per step for this calculation.)

e) Use the forward Euler method to compute the numerical solution of the above ordinary differential equation with same initial condition by hand for two steps with grid interval size $h = 1/10$.

f) The analytical solution is

$$y(x) = \left(\frac{125x+2}{2}\right)^{-2/5}$$

Using Matlab codes, indicate the maximum stable interval size possible for forward Euler method from the following: $h = 1/10$, $h = 1/15$, $h = 1/30$, $h = 1/45$, $h = 1/90$. How does your choice compare with the stability condition?

solution - (a)

Write Taylor series expansion for Y_{i+1} using node i as the base point

$$Y_{i+1} = Y_i + hY'(x_i, Y_i) + \frac{1}{2}h^2 Y''(x_i, Y_i) + \frac{1}{6}h^3 Y'''(x_i, Y_i) + \dots \frac{1}{n!}h^n Y^{(n)}(x_i, Y_i)$$

Solving for $Y'(x_i, Y_i)$

$$Y'(x_i, Y_i) = \frac{Y_{i+1} - Y_i}{h} - \frac{1}{2}hY''(x_i, Y_i) + O(h^2)$$

Therefore, truncation error is $\frac{1}{2}hY''(x_i, Y_i)$

As h goes to zero, the truncation error vanishes. Therefore, the method is consistent.

solution - (b)

The backward Euler method can be written as

$$Y_{i+1} = Y_i + hf(x_{i+1}, Y_{i+1})$$

solution - (c)

For the forward Euler method

$$Y_{i+1} = Y_i + hf(x_i, Y_i)$$

Considering $f(x, y) = -\lambda y$ gives

$$Y_{i+1} = Y_i + h(-\lambda Y_i)$$

$$= (1 - \lambda h)Y_i$$

Therefore, the amplification factor $G = 1 - \lambda h$

The method is stable if $|G| < 1$, which is

$$|1 - \lambda h| < 1$$

$$-1 < 1 - \lambda h < 1$$

The right-hand inequality $\lambda h > 0$ is always satisfied since λ is positive real constant.

The left-hand inequality $\lambda h < 2$ requires that $h < \frac{2}{\lambda}$

Therefore, the forward Euler method is conditionally stable. It is stable only when $h < \frac{2}{\lambda}$

Solution - (c) cont.

For the backward Euler method

$$Y_{i+1} = Y_i + h f(x_{i+1}, Y_{i+1})$$

Considering $f(x,y) = -\lambda y$ gives

$$Y_{i+1} = Y_i + h(-\lambda Y_{i+1})$$

$$= \frac{Y_i}{1+\lambda h}$$

Therefore, the amplification factor $G = \frac{1}{1+\lambda h}$

The method is stable if $|G| < 1$ which is true for all values of λh.

The backward Euler method is unconditionally stable.

Solution - (d)

Considering the ordinary differential equation

$$\frac{dy}{dx} = -25y^{3.5} \qquad \text{with } y(0) = 1$$

Using the backward Euler method with grid interval size $h = \frac{1}{10}$

Step 1 , $Y_1 = Y_0 + h(-25Y_1^{3.5})$

$$Y_1 = 1 - 2.5Y_1^{3.5}$$

Use 2 Newton iterations to solve for Y_1

In this step, $f(Y_1) = Y_1 + 2.5Y_1^{3.5} - 1 = 0$

$$f'(Y_1) = 1 + 8.75Y_1^{2.5}$$

$$Y_1^{(m+1)} = Y_1^{(m)} - \frac{f(Y_1^{(m)})}{f'(Y_1^{(m)})}$$

Let $Y_1^{(0)} = 1$. Use 2 Newton iterations

$$Y_1^{(1)} = 1 - \frac{(1+2.5(1)^{3.5} - 1)}{1+8.75(1)^{2.5}} = 0.74359$$

$$Y_1^{(2)} = 0.74359 - \frac{(0.74359 + 2.5(0.74359)^{3.5} - 1)}{1+8.75(0.74359)^{2.5}} = 0.62179$$

Solution - (d) cont.

Backward Euler method step 2

$$Y_2 = Y_1 + h(-25Y_2^{3.5})$$

$$Y_2 = 0.62179 - 2.5Y_2^{3.5}$$

Use 2 Newton iterations to solve for Y_2

In this step, $f(Y_2) = Y_2 + 2.5Y_2^{3.5} - 0.62179 = 0$

$$f'(Y_2) = 1 + 8.75Y_2^{2.5}$$

$$Y_2^{n+1} = Y_2^n + \frac{f(Y_2^n)}{f'(Y_2^n)}$$

Let $Y_2^{(0)} = 0.62179$. Use 2 Newton iterations.

$$Y_2^{(1)} = 0.62179 + \frac{(0.62179 + 2.5(0.62179)^{3.5} - 0.62179)}{1+8.75(0.62179)^{2.5}} = 0.49258$$

$$Y_2^{(2)} = 0.49258 + \frac{(0.49258 + 2.5(0.49258)^{3.5} - 0.62179)}{1+8.75(0.49258)^{2.5}} = 0.46025$$

The numerical solution of $\frac{dy}{dx} = -25y^{3.5}$ with $y(0) = 1$ using the backward Euler method for 2 steps with grid interval size $h = \frac{1}{10}$ is 0.46025

Solution - (e)

Considering $\frac{dy}{dx} = -25y^{3.5}$ with $y(0) = 1$ and using the forward Euler method with grid interval size $h = \frac{1}{10}$

step 1

$$Y_{i+1} = Y_i + hf(X_i, Y_i)$$

$$Y_1 = Y_0 + h(-25Y_0^{3.5})$$

$$= 1 - 2.5(1)^{3.5} = -1.5$$

step 2

$$Y_2 = Y_1 + h(-25Y_1^{3.5})$$

$$= -1.5 - 2.5(-1.5)^{3.5} = -1.5 + 10.33i$$

The numerical solution becomes complex number after 2 steps using the forward Euler method.

Solution - (f)

From Matlab codes, using the forward Euler method with $h = 1/15$ is stable but the solution is complex number while using $h = 1/30$ yields stable and real number solution.

Stability analysis can be performed only for linear differential equations.
Therefore, we need to linearize $f(y) = -25y^{3.5}$ locally using Taylor series

$$f(y) = f(a) + (y-a) f'(a) + \dots$$

$$-25y^{3.5} \approx -25a^{3.5} + (y-a)(-87.5a^{2.5})$$

Finding the stability limit for $\frac{dy}{dx} = -87.5a^{2.5}y + 62.5a^{3.5}$ using the forward Euler method

Comparing to the stability obtained in solution (c), the stability limit is $h < \frac{2}{87.5a^{2.5}}$

So the stability condition also depends on the value of "a" chose to linearize the function.